UNIVERSITY OF PITESTI
FACULTY OF MECHANICS AND TECHNOLOGY
SCIENTIFIC BULLETIN
AUTOMOTIVE series, year XV, no.19, vol. B

A FINITE ELEMENTS METHOD APPLICATION TO THE NONLINEAR MECHANICS BEHAVIOUR OF THREE DIMENSIONAL BEAMS

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Abstract: *Mechanical behavior of beams under large rotations and displacements was investigated. Using co-rotational approach given by Crisfield, three dimensional beam finite elements was modeled in Mathematica^{TM} environment. The yielding non-linear equation system was solved by utilizing Newton-Raphson technique. Dynamic balance equations and numerical time integration method were introduced, the solution process was left as a future work, however.*

Keywords: vehicle, finite elements,

INTRODUCTION

Large rotations and displacements must be taken into account in the three dimensional beam models used for the engineering applications, such as compliant mechanisms, robot arms, coil springs and space frames. Here, the non-vectorial nature of rotational variables complicates the nonlinear formulations [1]. A number of authors have introduced so-called co-rotational elements where the displacements coming from rotations and deformations are decomposed. In this formulation, each discerete element carries a Cartesian coordinate system that continuously rotates and translates with the element but does not deform with the element. Co-rotational formulation of this kind for modelling beam elements appear to have been first proposed by authors Oran and Kassimali [2] and Belytschko, *et al* [3]. Oran and Kassımali solved the dynamics problem of two dimensional beams by using co-rotational formulation which was called as 'beam-column' approach. They derived a consistent tangent stiffness matrix of beam element for small deformations but did not take into account large rotations. The nonlinear analysis of three dimensional beams using the co-rotational formulation where the corresponding tangent stiffness matrices derived for small deformations but large rotations was presented by some researchers [3-5]. Pai employed the co-rotational formulation for large deformations and rotations for three dimensional beams [6]. Research works including other formulations for three dimensional beams can be found in the literature [7-9].

EQUATIONS

Each node of the element has six degrees of freedom that consist of three rotations and three displacements (see Figure 1). Note that the subscript "*l*" represents local variables. Here, the element behavior is linear in local coordinates however nonlinear behavior emerges from the co-rotational formulation for large rotations.
Local degrees of freedom for the element are;

$$\mathbf{P}_l^T = (\mathbf{d}_{l1}^{\mathrm{T}}, \boldsymbol{\theta}_{l1}^{\mathrm{T}}, \mathbf{d}_{l2}^{\mathrm{T}}, \boldsymbol{\theta}_{l2}^{\mathrm{T}}) \tag{1}$$

$$\mathbf{d}_l^T = (u_{l1}, v_{l1}, w_{l2}) \tag{2}$$

The unit vector $\mathbf{e}_1$ passes between first node and second node and can be found as;

$$\mathbf{e}_1 = (\mathbf{x}_{21} + \mathbf{d}_{21})/l_2 \tag{3}$$

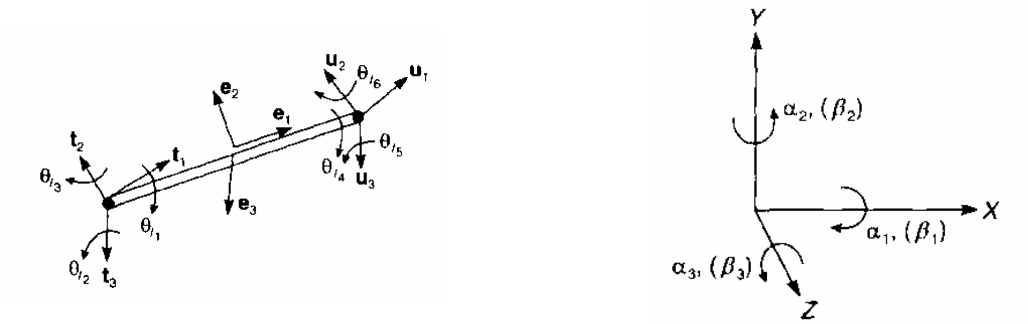

Figure 1: Beam element and degrees of freedom.

A linear relationship can be written between local nodal forces ($\mathbf{q}_l$) and rotations ($\mathbf{p}_l$) using Euler-Bernoulli beam assumptions.

$$\mathbf{q}_l = \mathbf{K}_l \mathbf{p}_l \tag{4}$$

Axial forces and moments depend on the local strains and rotations as;

$$N = \frac{EA}{l_0} u_l, \quad \mathbf{M} = \mathbf{D}(\boldsymbol{\theta}_l - \boldsymbol{\theta}_{l0}) \tag{5}$$

The subscript zero denotes the initial configutation and **D** is the stiffness matrix for the moment-rotation relation. Consider the relation between local and global degrees of freedom as $\delta \mathbf{p}_l = \mathbf{F}^{\mathrm{T}} \delta \mathbf{p}$ and with the help of the virtual work principle global internal force vector will be as follows;

$$\mathbf{q}_i = \mathbf{F}^{\mathrm{T}} \mathbf{q}_{li} = \mathbf{F}^{\mathrm{T}} \mathbf{K}_l \mathbf{p}_l \,. \tag{6}$$

Then the tangent stiffness matrix can be written as;

$$\delta \mathbf{q}_i = \mathbf{F}^{\mathrm{T}} \delta \mathbf{q}_{li} + \delta \mathbf{F}^{\mathrm{T}} \mathbf{q}_{li} = \mathbf{F}^{\mathrm{T}} \mathbf{K}_l \mathbf{F} \delta \mathbf{p} \quad + \mathbf{K}_{t\sigma} \delta \mathbf{p} \quad . \tag{7}$$

Local base vectors defined on an element can be seen in Fig.2. Here, the rotation matrices **E**, **U** and **T** are composed of corresponding base vectors aligned in columns.

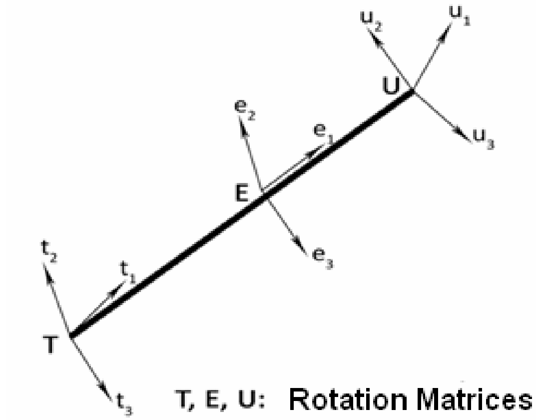

Figure 2: Current base vectors and rotation matrices

The relation between the global internal force vector and displacements can be written as;

$$\delta \mathbf{q}_i = \mathbf{K}_t \, \delta \mathbf{p} \tag{8}$$

Here, $\mathbf{K}_t$ denotes for the tangent stiffness matrix. Calculation of $\mathbf{K}_t$ is so cumbersome that the detailed procedure how to obtain $\mathbf{K}_t$ and the rotation matrices can be found in [4].

The tangent stiffness matrix for this formulation is not symmetric, but numerical experiments have shown that it becomes symmetric as the iterative procedure reaches equilibrium. Here, Newton-Raphson method is utilized for the iterative procedure to solve the nonlinear equation system. Below the algorithm for the solution process is summarized.

Steps:

1. At the first step; form the elements and calculate the initial rotation matrices (**T**, **E**, **U**).
2. Form the external force vector ($\mathbf{q_e}$).
3. Calculate the rotation matrices for the current load step.
4. Form the element tangent stiffness matrices ($\mathbf{K}_l$).
5. Assemble the global tangent stiffness matrix ($\mathbf{K}_t$).
6. Apply the boundary conditions.
7. Inverse the $\mathbf{K}_t$.
8. Calculate displacement increment (Eq. 8).
9. Update the nodal coordinates.
10. Check for the convergence; if test is satisfied continue, if test is not satistied go to step 3.
11. Obtain the displacement data.

Dynamic behavior of the beam can be analyzed by using any time integration method. In general, equation of balance for the system at the end of a time step will be

$$(\mathbf{q}_{i,n+1} - \mathbf{q}_{e,n+1}) + \mathbf{M}\ddot{\mathbf{d}}_{n+1} + \mathbf{C}\dot{\mathbf{d}}_{n+1} = \mathbf{0}\,. \tag{9}$$

In this equation, subscripts *i* and *e* denotes internal and external vectors respectively, **M** is mass matrix, **C** is damping matrix and **d** is displacement vector. Making use of Newmark-β time integration method, displacement and velocity vectors for each time step will be as follows;

$$\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t\dot{\mathbf{d}}_n + \Delta t^2((1-2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1})/2 \tag{10}$$

$$\dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \Delta t((1-\gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1} \tag{11}$$

The above equations are substituted into the general dynamic balance equation and employing the tangent stiffness matrix;

$$\mathbf{q}_{i,n} - \mathbf{q}_{e,n+1} + \mathbf{K}_{t,n}\Delta\mathbf{d} + \mathbf{M}(\tfrac{4}{\Delta t^2}\Delta\mathbf{d} - \tfrac{4}{\Delta t}\dot{\mathbf{d}}_n - \ddot{\mathbf{d}}_n) + \mathbf{C}(\tfrac{2}{\Delta t}\Delta\mathbf{d} - \dot{\mathbf{d}}_n) = \mathbf{0} \tag{12}$$

Hence, at each time step these nonlinear equations are solved by using Newton-Raphson method. However, the dynamic problems are left as a future work and will not be covered in this study and only solutions for static problems will be given below.

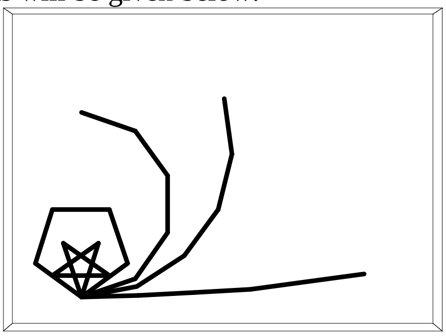

Figure 3: Cantilever subject to end moment. Applied moment values are 0.025M*, 0.3M* , 0.5M*, M*, 2M* (from right to left). $M^* = \frac{M\,L}{2\pi EI}$

NUMERICAL EXAMPLES

Exampe 1: Cantilever subject to an end moment

A straight cantilever beam is subject to a bending moment at the right end (see Fig.3). This is actually a two dimensional pure bending problem, but is solved using the current formulation to varify the results. Here, 5 elements are modeled and all solution process is accomplished in *Mathematica*^{TM} [10] environment. A convergence criterion for Newton-Raphson method is chosen as $\epsilon = \frac{\|\mathbf{g}\|}{\|\mathbf{q}_e\|} < 1/1000$ and only 4 iterations are found to be enough at each time step. Here, **g** is for the out of balanced forces and $\mathbf{q}_e$ is for the total external forces.

Example 2: Forty five degree bend

A cantilever arc is loaded by a single end force applied perpendicular to the arc plane (see Fig. 4). Elasticity modulus is taken as 10^7 and the arc is modelled by 8 elements. Maximum load value is 600 and it is divided into 8 equal load steps and 5 iterations for each step are required to converge. The results are shown in Fig.5.

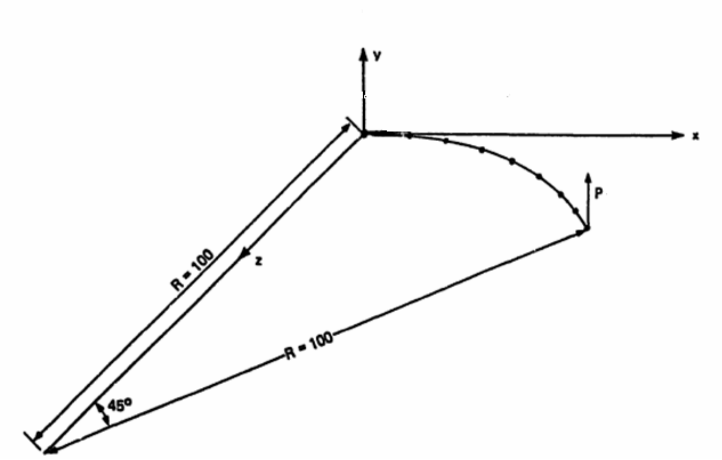

Figure 4: Forty five degree bend geometry.

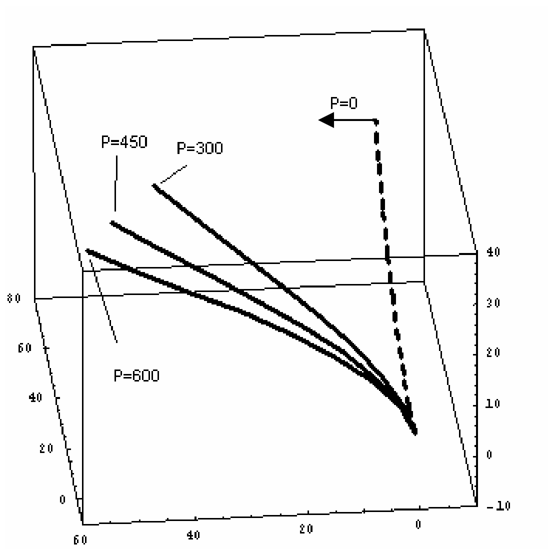

Figure 5: Deformed shapes of 45^0 bend.

Example 3: Clamped one-turn coil spring

One end (down) clamped and the other end is pulled upward, helical spring has been studied by using 20 elements for linear and nonlinear behavior (see Fig.6). Inhere, Young's modulus is 10^7, helical radius R=10, spring radius r = 1, helical slope 10 degrees and force is 100 in nondimensional units. The displacements of the end of the spring:

Linear solution: u_x= -0.821422, u_y= 0.37414, u_z= 3.12548

Nonlinear solution: u_x=-0.921413, u_y=0.35634, u_z=2.97771

In nonlinear solution process, the force is applied at one step and converge is achieved after 7 iterations. The deformed configuration is shown in Figure 6.

CONCLUSIONS

Mechanical behavior of beams under large rotations and displacements was investigated. Using co-rotational approach given by Crisfield, three dimensional beam finite elements was modeled in *Mathematica*^{TM} environment. The yielding non-linear equation system was solved by utilizing

Newton-Raphson technique. The solutions obtained from examples are compared to be consistent with those of Crisfield's. There are some situations where a structure is subjected to small deformation changes, but large rotations (i.e., springs). The formulation derived here might be useful for such cases. Dynamic balance equations and numerical time integration method were introduced, the solution process was left as a future work, however.

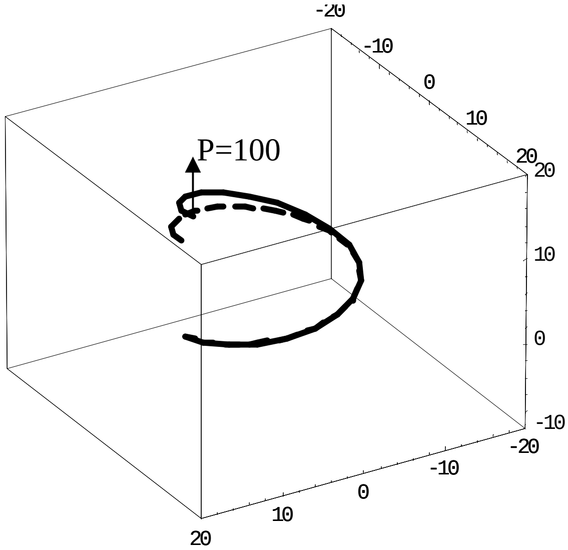

Figure 6: Deformed shape of one turn coil spring.

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