UNIVERSITY OF PITESTI
FACULTY OF MECHANICS AND TECHNOLOGY
SCIENTIFIC BULLETIN
AUTOMOTIVE series, year XV, no.19, vol. B

THE OPTIMAL DESIGN OF LINKAGE BASED ON MATLAB

Ion ION
University of Pitesti, Romania,

Abstract: *Optimal problem is often met in engineering practice. As the coupler-link of linkages can produce useful motions that are desired in many engineering applications, research on coupler-link motion has attracted the attention of various investigators for over a century. The basic theme of this paper is the kinematic optimization of six-bar linkage -with the assumption that all the links are rigid- for the problem of function generation. The optimization methodology developed here is based upon a minimization of the linkage errors defined as the difference between the prescribed coupler-point (positions, velocities, accelerations) and the actual ones attained by the linkage.*

Keywords: linkage, optimization, kinematic analysis, constraint conditions, objective function.

INTRODUCTION

Optimization is the act of obtaining the best result under given circumstances. Optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function similar to minimize the effort required or to maximize the desired benefit. The classical methods of optimization are useful in finding the optimum solution of continuous and differentiable functions. These methods are analytical and make use of the techniques of differential calculus in locating the optimum points. Since some of the practical problems involve objective functions that are not continuous and/differentiable, the classical optimization techniques have limited scope in practical applications. In this paper we present a study of a multivariable function with equality and inequality constraints for a planar linkage. Linkages are an important class of mechanisms. Unlike gear or cam mechanisms, linkages transform motion from one rigid body to another through surface contact between every pair of adjacent links. All the joints used in construction linkages are lower kinematic pairs, i.e., surface-contact couplings between two rigid bodies. Depending on the type of joints a linkage contains, it can perform planar or spatial motions.

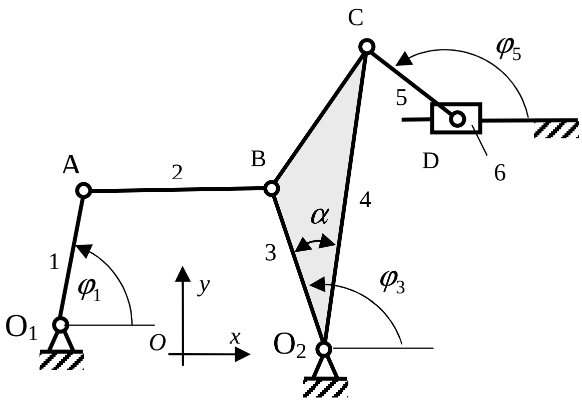

Fig. 1. Planar six-bar mechanism

KINEMATIC ANALYSSIS OF LINKAGE

The considered six-bar planar mechanism is shown in Fig. 1. The driver link is the rigid link 1 and the origin of the reference frame is at O. The motion of coupler point D is defined when the position vector, velocity and acceleration of this point are defined as function of time with respect to a fixed reference frame with the origin at O.

The coordinates of joint A are expressed in terms of the coordinates of joint O_1 and the relative orientation of link 1. Its coordinates are determined using following equations

$$x_A = x_{O1} + l_1 \cos\varphi_1 \tag{1}$$

$$y_A = y_{O1} + l_1 \sin\varphi_1 \tag{2}$$

For the joint B on the links 2 and 3 can be writing

$$x_B = x_A + l_2 \cos\varphi_2 = x_{O2} + l_3 \cos\varphi_3 \tag{3}$$

$$y_B = y_A + l_2 \sin\varphi_2 = x_{O2} + l_3 \sin\varphi_3 \tag{4}$$

where φ_2 and φ_3 are the relative orientations of links 2 and 3 with respect to axis ox of the Cartesian reference frame *Oxy*.

By eliminating φ_4 combining equation (3) with (4) and summarizing we have

$$\left[(x_A - x_{O2}) + l_2 \cos\varphi_2\right]^2 + \left[(y_A - y_{O2}) + l_2 \sin\varphi_2\right]^2 = l_3^2 \tag{5}$$

or

$$A\cos\varphi_2 + B\sin\varphi_2 + C = 0 \tag{6}$$

Where

$$A = 2l_2(x_A - x_{O2}) \tag{7}$$

$$B = 2l_2(y_A - y_{O2}) \tag{8}$$

$$C = (x_A - x_{O2})^2 + (y_A - y_{O2})^2 + l_2^2 - l_3^2 \tag{9}$$

With notation

$$T = \tan\frac{\varphi_2}{2} \tag{10}$$

equation (6) can be writing as

$$(C - A)T^2 + 2BT + C + A = 0 \tag{11}$$

and is obtained

$$T = \frac{-B \pm \sqrt{B^2 - C^2 + A^2}}{C - A} \tag{12}$$

respective

$$\varphi_2 = 2\arctan T \tag{13}$$

In the other hand the position of joint C is

$$x_C = x_{O2} + l_4 \cos(\varphi_3 - \alpha) \tag{14}$$

$$y_C = y_{O2} + l_4 \sin(\varphi_3 - \alpha) \tag{15}$$

For the joint D, between the links 6 and 7 can be writing

$$y_D = y_C - l_5 \sin\varphi_5 = e \tag{16}$$

and result

$$\varphi_5 = \arcsin\frac{y_C - e}{l_5} \tag{17}$$

$$x_D = x_C - l_5 \cos\varphi_5 \tag{18}$$

The velocities of coupler points A, B, C and D can be written as form

$$\mathbf{v}_A = \omega_1 \times O_1A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_A - x_{O1} & y_A - y_{O1} & 0 \end{vmatrix} = \begin{vmatrix} -l_1\omega_1 \sin\varphi_1 \\ l_1\omega_1 \cos\varphi_1 \\ 0 \end{vmatrix} \tag{19}$$

$$\mathbf{v}_B = \omega_3 \times O_2B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_B - x_{O2} & y_B - y_{O2} & 0 \end{vmatrix} = \begin{vmatrix} -l_3\omega_3 \sin\varphi_3 \\ l_3\omega_3 \cos\varphi_3 \\ 0 \end{vmatrix} \tag{20}$$

$$\mathbf{v}_C = \omega_3 \times O_2C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_C - x_{O2} & y_C - y_{O2} & 0 \end{vmatrix} = \begin{vmatrix} -l_4\omega_3 \sin(\varphi_3 - \alpha) \\ l_4\omega_3 \cos(\varphi_3 - \alpha) \\ 0 \end{vmatrix} \tag{21}$$

$$\mathbf{v}_D = \mathbf{v}_C + \omega_5 \times CD = \begin{vmatrix} -l_4\omega_3 \sin(\varphi_3 - \alpha) + l_5\omega_5 \sin\varphi_5 \\ l_4\omega_3 \cos(\varphi_3 - \alpha) - l_5\omega_5 \cos\varphi_5 \\ 0 \end{vmatrix} = \begin{vmatrix} -l_4\omega_3 \sin(\varphi_3 - \alpha) + l_5\omega_5 \sin\varphi_5 \\ 0 \\ 0 \end{vmatrix} \tag{22}$$

where the angular velocities of links 2, 3 and 5 are forms

$$\omega_2 = \omega_1 \frac{l_1 \sin(\varphi_1 - \varphi_3)}{l_2 \sin(\varphi_3 - \varphi_2)} \tag{23}$$

$$\omega_3 = \omega_1 \frac{l_1 \sin(\varphi_1 - \varphi_2)}{l_3 \sin(\varphi_3 - \varphi_2)} = \omega_4 \tag{24}$$

$$\omega_5 = \omega_3 \frac{l_4 \cos(\varphi_3 - \alpha)}{l_5 \cos\varphi_5} \tag{25}$$

The angular accelerations of links 2, 3 and 5 are forms

$$\varepsilon_2 = \omega_1 l_1 \frac{(\omega_1 - \omega_3)\cos(\varphi_1 - \varphi_3)\sin(\varphi_3 - \varphi_2) - (\omega_3 - \omega_2)\sin(\varphi_1 - \varphi_3)\cos(\varphi_3 - \varphi_2)}{l_2(\sin(\varphi_3 - \varphi_2))^2} \tag{26}$$

$$\varepsilon_3 = \omega_1 l_1 \frac{(\omega_1 - \omega_2)\cos(\varphi_1 - \varphi_2)\sin(\varphi_3 - \varphi_2) - (\omega_3 - \omega_2)\sin(\varphi_1 - \varphi_2)\cos(\varphi_3 - \varphi_2)}{l_3(\sin(\varphi_3 - \varphi_2))^2} \tag{27}$$

$$\varepsilon_5 = \frac{l_4}{l_5}\left[\varepsilon_3 \frac{\cos(\varphi_3 - \alpha)}{\cos\varphi_5} - \omega_3 \frac{\omega_3 \cos\varphi_5 \sin(\varphi_3 - \alpha) - \omega_5 \sin\varphi_5 \cos(\varphi_3 - \alpha)}{(\cos\varphi_5)^2}\right] \tag{28}$$

The acceleration of coupler point D is

$$a_D = -l_4\left[\varepsilon_3 \sin(\varphi_3 - \alpha) + \omega_3^2 \cos(\varphi_3 - \alpha)\right] + l_5\left[\varepsilon_5 \sin\varphi_5 + \omega_5^2 \cos\varphi_5\right] \tag{29}$$

In the optimization process all equality and inequality constraints and conditions should be considered. The first inequality constraints comes from that there is no negative length

$$l_1, l_2, ..., l_5 > 0 \tag{30}$$

respective

$$B^2 - C^2 + A^2 \geq 0 \tag{31}$$

OPTIMIZATION ALGORITHM

In the synthesis of function generators, the linkage sought is to produce a set of input and output variable pair that verifies certain functional relations. Although it is not always possible to find a linkage that generates a given set input-output pair exactly, an optimum linkage can always be found that produces the given pairs approximately, with a minimum error. Optimization bears more practical impact than the conventional synthesis because of its ability to handle unlimited numbers of given input-output pairs and meet many design requirements at the same time. Thus, through optimization, various design conditions can be satisfied and globally meaningful solutions can be obtained.

Since the mobility conditions and nonlinear inequality constraints a constrained nonlinear optimization method based on *fmincon* function is proposed to solve this problem. This method is very efficient in handling the function-generator synthesis problems, regardless of the linkage type, i.e., planar, spherical or spatial.

Optimization techniques are used to find a set of design parameters, $x = \{x_1, x_1, ..., x_n\}$, that can in some way be defined as optimal. In a simple case this might be the minimization or maximization of some system characteristic that is dependent on x. In a more advanced formulation the objective function, $f(x)$, to be minimized or maximized

$$\min_{x} f(x) \tag{32}$$

might be subject to constraints in the form of equality constraints

$$G_i(x) = 0 \quad (i = 1,\ldots,m_e) \tag{33}$$

inequality constraints

$$G_i(x) \le 0 \quad (i = m_e + 1,\ldots,m) \tag{34}$$

and/or parameter bounds, x_l, x_u.

The objective function $f(x)$ returns a scalar value, and the vector function $G(x)$ returns a vector of length m containing the values of the equality and inequality constraints evaluated at x.

Many of the methods used in Matlab/Optimization Toolbox solvers are based on *active set algorithm*, a simple yet powerful concept in optimization.

In constrained optimization, the general aim is to transform the problem into an easier sub-problem that can then be solved and used as the basis of an iterative process. A characteristic of a large class of early methods is the translation of the constrained problem to a basic unconstrained problem by using a penalty function for constraints that are near or beyond the constraint boundary. In this way the constrained problem is solved using a sequence of parameterized unconstrained optimizations, which in the limit (of the sequence) converge to the constrained problem. These methods are now considered relatively inefficient and have been replaced by methods that have focused on the solution of the Karush-Kuhn-Tucker (KKT) equations. The KKT equations are necessary conditions for optimality for a constrained optimization problem. If the problem is a so-called convex programming problem, that is, $f(x)$ and $G_i(x)$, are convex functions, then the KKT equations are both necessary and sufficient for a global solution point.

Referring to (32), the Kuhn-Tucker equations can be stated as

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \cdot \nabla G_i(x^*) = 0 \tag{35}$$

$$\lambda_i \cdot \nabla G_i(x^*) = 0 \quad (i = 1,\ldots,m_e) \tag{36}$$

$$\lambda_i > 0 \quad (i = m_e + 1,\ldots,m) \tag{37}$$

in addition to the original constraints (33) and (34).

The equation (35) describes a canceling of the gradients between the objective function and the active constraints at the solution point. For the gradients to be canceled, Lagrange multipliers λ_i are necessary to balance the deviations in magnitude of the objective function and constraint gradients. Because only active constraints are included in this canceling operation, constraints that are not active must not be included in this operation and so are given Lagrange multipliers equal to 0. This is stated implicitly in the last two Kuhn-Tucker equations.

The solution of the KKT equations forms the basis to many nonlinear programming algorithms. These algorithms attempt to compute the Lagrange multipliers directly. Constrained quasi-Newton methods guarantee super-linear convergence by accumulating second-order information regarding the KKT equations using a quasi-Newton updating procedure. These methods are commonly referred to as Sequential Quadratic Programming (SQP) methods, since a QP sub-problem is solved at each major iteration (also known as Iterative Quadratic Programming, Recursive Quadratic Programming, and Constrained Variable Metric methods).

In this case are used a set of design parameters $\mathbf{x} = [x_1, x_2, x_3, \ldots x_{11}]^T$, defined as: $x_1 = x_{O1}$; $x_2 = y_{O1}$; $x_3 = x_{O2}$; $x_4 = y_{O2}$; $x_5 = l_1$; $x_6 = l_2$; $x_7 = l_3$; $x_8 = l_4$; $x_9 = l_5$; $x_{10} = e$; $x_{11} = \alpha$.

The objective function is defined as

$$f(x) = (x_D - x_{D-desired})^2 + (v_D - v_{D-desired})^2 + (a_D - a_{D-desired})^2$$

and the constraints (equations 30, 31) can be written as

$$-x_5 \le 0;\ -x_6 \le 0;\ -x_7 \le 0;\ -x_8 \le 0;\ -x_9 \le 0$$

$$-[2x_6(x_2 + x_5 \sin\varphi_1 - x_4)]^2 + \left[(x_1 + x_5 \cos\varphi_1 - x_3)^2 + (x_2 + x_5 \sin\varphi_1 - x_4)^2 + x_6^2 - x_7^2\right]^2 -$$
$$-[2x_6(x_1 + x_5 \cos\varphi_1 - x_3)]^2 \le 0$$

NUMERICAL EXAMPLE

A design problem is solved here to demonstrate the application of this optimization method. It is required to design a planar path generation mechanism to meet the input-output relations shown in table 1.

$\varphi_1 = 10^0$	$\varphi_1 = 30^0$	$\varphi_1 = 60^0$
$x_{D-desired} = 5\,[\mathrm{cm}]$	$v_{D-desired} = 3\,[\mathrm{cm/s}]$	$a_{D-desired} = -7\,[\mathrm{cm/s^2}]$

Table. 1. Desired input-output relations

The problem is solved using nonlinear inequality constraints by the fmincon function and obtained the solution

$$\mathbf{x} = [14.335, 13.339, 51.621, 1.008, 0.632, 41.645, 66.245, 54.609, 44.963, 5.654, -0.022]^T$$

The linkage parameters of planar six-bar linkages are defined as:

$$x_{O1} = 14.335[cm];\ y_{O1} = 13.339[cm];\ x_{O2} = 51.621[cm];\ y_{O2} = 1.008[cm];\ l_1 = 0.632[cm];$$
$$l_2 = 41.645[cm];\ l_3 = 66.245[cm];\ l_4 = 54.609[cm];\ l_5 = 44.963[cm];\ e = 5.654[cm];$$
$$\alpha = -0.022[rad].$$

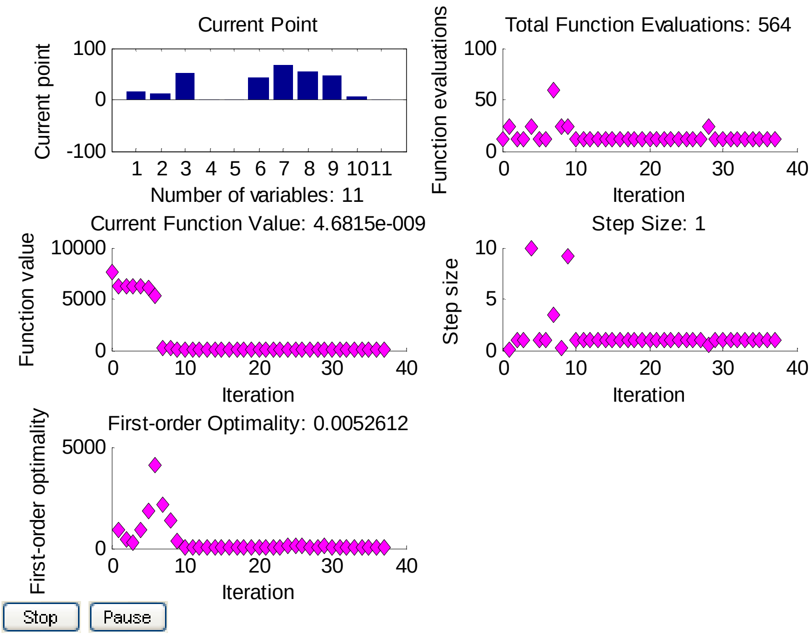

Fig. 2. Optimization plot function by Matlab/Optimization tool

CONCLUSIONS

In this paper one of the basic problems in linkage kinematics, input-output analysis, is presented. The aim of this paper is to establish the basic context of function-generating linkage synthesis problems. The basic concepts for path generation are discussed to support the formulations presented in this work. The optimization scheme presented above is very efficient in solving the problems of synthesis of linkages under mobility constraints. This scheme is demonstrated by minimizing the design error. The optimization procedure is illustrated with a numerical example. Optimal synthesis of linkage is, in fact, a repeated analysis for a random determined mechanism and finding of the best possible one so that it could meet technological requirements, and it is most often used in dimensional synthesis, which implies determination of elements of the given mechanism (lengths, angles, coordinates) necessary for creation of the linkage in the direction of desired motion. With same specific modifications, this optimization model can be extended to other mechanical systems.

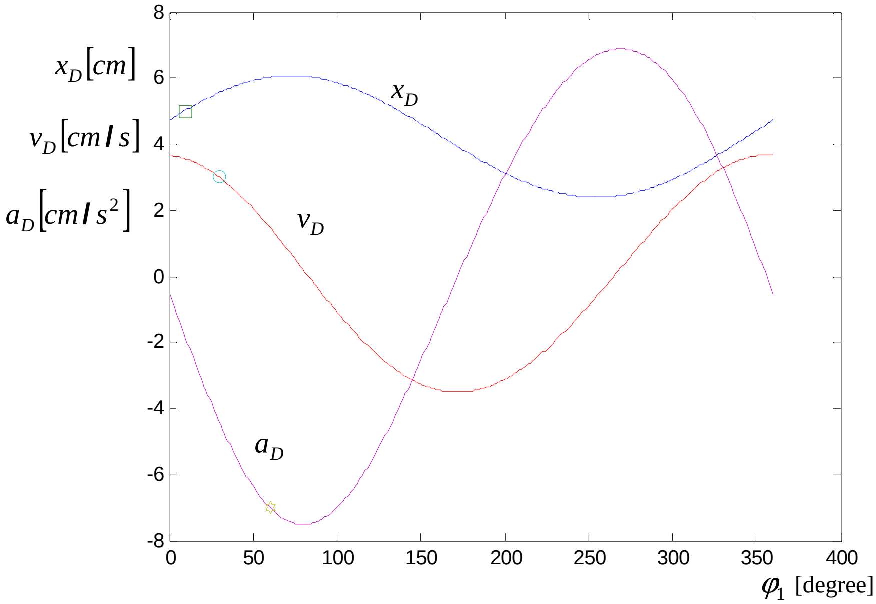

Fig. 3. The optimum coupler-point curves of linkage

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