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DYNAMICS OF AN AUTOMOBILE WHEEL

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Abstract: *In our paper we study the dynamics of an automobile wheel. The wheel is considered as a torus one. We obtained the equations of motion in multibody form. A numerical application is also considered, the diagrams obtained by numerical simulations being presented.*

Keywords: torus, wheel, multibody

INTRODUCTION

In the scientific literature is considered that the automobile wheel is a circular disk, rolling without sliding on a horizontal plan. In this paper we shall make a step forward considering that the wheel is a torus, which is the real case. The wheel has six degrees of freedom, three translations and three rotations. The reaction has three components along the three axes of coordinates. The equations of motion are obtained using the multibody theory that is we use the Lagrange second order equations and the matrix of constraints. The equations are more complicate, but they reflect in a more accurate form the real case. We consider the both cases with and without rolling friction.

NOTATIONS

The following notations are used
– m, the mass of the wheel;
– $[\mathbf{m}]$, the matrix defined by

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}; \tag{1}$$

– J_x, J_y, J_z, the principal, central moments of inertia;
– $[\mathbf{J}]$, the matrix given by

$$[\mathbf{J}] = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}; \tag{2}$$

– O, the centre of the torus;
– r_0, r, the radii of the torus;
– s, the coefficient of the rolling friction;
– ψ, θ, φ, the angles of Euler;
– $[\boldsymbol{\psi}]$, $[\boldsymbol{\theta}]$, $[\boldsymbol{\varphi}]$, the matrices

$$[\boldsymbol{\psi}] = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},\ [\boldsymbol{\theta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},\ [\boldsymbol{\varphi}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{3}$$

– $[\mathbf{A}]$, the rotational matrix

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}]; \tag{4}$$

– $[\mathbf{U}_\psi]$, $[\mathbf{U}_\theta]$, $[\mathbf{U}_\varphi]$, the matrices

$$[\mathbf{U}_\psi]=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix},\ [\mathbf{U}_\theta]=\begin{bmatrix}0&0&0\\0&0&-1\\1&0&0\end{bmatrix},\ [\mathbf{U}_\varphi]=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}; \tag{5}$$

– $[\mathbf{A}_\psi]$, $[\mathbf{A}_\theta]$, $[\mathbf{A}_\varphi]$, the partial derivatives of the rotational matrix

$$[\mathbf{A}_\psi]=[\mathbf{U}_\psi][\mathbf{A}],\ [\mathbf{A}_\theta]=[\mathbf{A}][\boldsymbol{\varphi}]^T[\mathbf{U}_\theta][\boldsymbol{\varphi}],\ [\mathbf{A}_\varphi]=[\mathbf{A}][\mathbf{U}_\varphi]; \tag{6}$$

– $[\dot{\mathbf{A}}]$, the derivatives of the matrix $[\mathbf{A}]$ with respect to time

$$[\dot{\mathbf{A}}]=\dot{\psi}[\mathbf{A}_\psi]+\dot{\theta}[\mathbf{A}_\theta]+\dot{\varphi}[\mathbf{A}_\varphi]; \tag{7}$$

– $[\mathbf{Q}]$, the matrix given by

$$[\mathbf{Q}]=\begin{bmatrix}\sin\theta\sin\varphi & \cos\varphi & 0\\ \sin\theta\cos\varphi & -\sin\varphi & 0\\ \cos\theta & 0 & 1\end{bmatrix}; \tag{8}$$

– $[\mathbf{Q}_\theta]$, $[\mathbf{Q}_\varphi]$, the partial derivatives of the matrix $[\mathbf{Q}]$

$$[\mathbf{Q}_\theta]=\begin{bmatrix}\cos\theta\sin\varphi & 0 & 0\\ \cos\theta\cos\varphi & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ [\mathbf{Q}_\varphi]=\begin{bmatrix}\sin\theta\cos\varphi & -\sin\varphi & 0\\ -\sin\theta\sin\varphi & -\cos\varphi & 0\\ 0 & 0 & 0\end{bmatrix}; \tag{9}$$

– $[\dot{\mathbf{Q}}]$, the derivative of the matrix $[\mathbf{Q}]$ with respect to time

$$[\dot{\mathbf{Q}}]=\dot{\theta}[\mathbf{Q}_\theta]+\dot{\varphi}[\mathbf{Q}_\varphi]; \tag{10}$$

– $[\mathbf{r}]$, the matrix

$$[\mathbf{r}]=\begin{bmatrix}0 & r\cos\theta & -(r_0+r\sin\theta)\cos\varphi\\ -r\cos\theta & 0 & (r_0+r\sin\theta)\sin\varphi\\ (r_0+r\sin\theta)\cos\varphi & -(r_0+r\sin\theta)\sin\varphi & 0\end{bmatrix}; \tag{11}$$

– $[\mathbf{r}_\theta]$, $[\mathbf{r}_\varphi]$, the partial derivatives of the matrix $[\mathbf{r}]$

$$[\mathbf{r}_\theta]=\begin{bmatrix}0 & -r\sin\theta & -r\cos\theta\cos\varphi\\ r\sin\theta & 0 & r\cos\theta\sin\varphi\\ r\cos\theta\cos\varphi & -r\cos\theta\sin\varphi & 0\end{bmatrix},$$

$$[\mathbf{r}_\varphi]=\begin{bmatrix}0 & 0 & (r_0+r\sin\theta)\sin\varphi\\ 0 & 0 & (r_0+r\sin\theta)\cos\varphi\\ -(r_0+r\sin\theta)\sin\varphi & -(r_0+r\sin\theta)\cos\varphi & 0\end{bmatrix}; \tag{12}$$

– $[\dot{\mathbf{r}}]$, the derivative of the matrix $[\mathbf{r}]$ with respect to time

$$[\dot{\mathbf{r}}]=\dot{\theta}[\mathbf{r}_\theta]+\dot{\varphi}[\mathbf{r}_\varphi]; \tag{13}$$

– $[\mathbf{I}]$, the third order unity matrix

$$[\mathbf{I}]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}; \tag{14}$$

– $\{\mathbf{F}\}$, the column matrix of the external forces

$$\{\mathbf{F}\}=[0\ \ 0\ \ -mg\ \ 0\ \ 0\ \ 0]^T; \tag{15}$$

– $\{\tilde{\mathbf{F}}_\beta\}$, the column matrix

$$\{\tilde{\mathbf{F}}_\beta\}=\left[[\mathbf{J}][\dot{\mathbf{Q}}]+[\mathbf{A}]^T[\dot{\mathbf{A}}][\mathbf{J}][\mathbf{Q}]\right]\begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\varphi}\end{bmatrix}; \tag{16}$$

– $\{\tilde{\mathbf{F}}\}$, the column matrix

$$\{\tilde{\mathbf{F}}\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \{\tilde{\mathbf{F}}_{\beta}\} \end{bmatrix}; \tag{17}$$

– $[\mathbf{M}]$, the matrix of inertia

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{m}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{J}][\mathbf{Q}] \end{bmatrix}. \tag{18}$$

EQUATIONS OF MOTION

Following [4], one obtains the matrix of constraints

$$[\mathbf{B}] = \left[[\mathbf{I}] \;\; [\mathbf{A}][\mathbf{r}]^T [\mathbf{Q}] \right], \tag{19}$$

its derivative with respect to time being

$$[\dot{\mathbf{B}}] = \left[[\mathbf{0}] \;\; [\dot{\mathbf{A}}][\mathbf{r}]^T [\mathbf{Q}] + [\mathbf{A}][\dot{\mathbf{r}}]^T [\mathbf{Q}] + [\mathbf{A}][\mathbf{r}]^T [\dot{\mathbf{Q}}] \right]. \tag{20}$$

Knowing that the component ω_z of the vector $\boldsymbol{\omega}$ is

$$\omega_z = [0 \;\; 0 \;\; 1][\mathbf{Q}] \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} \tag{21}$$

and denoting by $[\mathbf{B}^*]$ and $[\tilde{\mathbf{B}}]$ the matrices

$$[\mathbf{B}^*] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -s\dfrac{\omega_z}{|\omega_z|} \end{bmatrix}, \tag{22}$$

$$[\tilde{\mathbf{B}}] = \begin{bmatrix} [\mathbf{I}] \\ [\mathbf{r}][\mathbf{A}]^T + [\mathbf{B}^*] \end{bmatrix}, \tag{23}$$

we deduce the metrical equation of motion

$$\begin{bmatrix} [\mathbf{M}] & -[\tilde{\mathbf{B}}] \\ [\mathbf{B}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\lambda\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}\} + \{\tilde{\mathbf{F}}\} \\ -[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{24}$$

where

$$\{\mathbf{q}\} = [X_O \;\; Y_O \;\; Z_O \;\; \psi \;\; \theta \;\; \varphi]^T \tag{25}$$

and $\{\boldsymbol{\lambda}\}$ are the Lagrange multipliers

$$\{\boldsymbol{\lambda}\} = [\lambda_1 \;\; \lambda_2 \;\; \lambda_3]^T. \tag{26}$$

APPLICATION

Let us consider the case described by (r_0 is the radius of circle on which is the torus, r defines the torus)

$$r_0 = 0.3\,\text{m}, \; r = 0.05\,\text{m}, \tag{27}$$

$$s^* = 0 \text{ (no rolling friction) or } s^* = 1 \text{ (rolling friction)}, \tag{28}$$

$$s = \frac{r_0}{10} s^*, \; s = 0\,\text{m} \text{ (no rolling friction)}, \; s = 0.03\,\text{m} \text{ (rolling friction)}, \tag{29}$$

$$m = 20\,\text{kg}, \tag{30}$$

$$J_x = \frac{mr_0^2}{2} = 0.9\,\mathrm{kgm}^2,\ J_y = \frac{mr_0^2}{2} = 0.9\,\mathrm{kgm}^2,\ J_z \approx mr_0^2 = 1.8\,\mathrm{kgm}^2. \tag{31}$$

The initial conditions are at $t = 0\,\mathrm{s}$

$$\psi = \frac{\pi}{2}\,\mathrm{rad},\ \theta = \frac{5\pi}{12}\,\mathrm{rad},\ \varphi = 0\,\mathrm{rad},\ X_O = -r_0\cos\theta = -0.07765\,\mathrm{m},\ Y_O = 0\,\mathrm{m},$$
$$Z_O = r_0\sin\theta + r = 0.33978\,\mathrm{m}, \tag{32}$$

$$\dot\psi = 0\,\mathrm{rad/s},\ \dot\theta = 0\,\mathrm{rad/s},\ \dot\varphi = -2\,\mathrm{rad/s},\ \dot X_O = 0\,\mathrm{m/s},\ \dot Y_O = -(r_0 + r\sin\theta)\dot\varphi = 0.69659\,\mathrm{m/s},$$
$$\dot Z_O = 0\,\mathrm{m/s}. \tag{33}$$

The diagrams are captured in the following figures. In the first three figures we considered the case characterized by no rolling friction, and in the last three diagrams is presented the case characterized by the existence of the rolling friction.

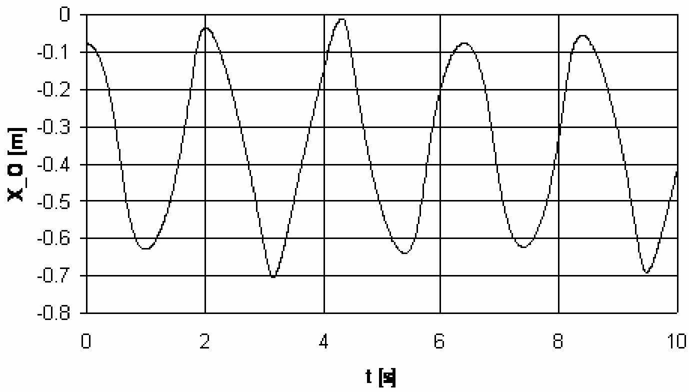

Fig. 1. The variation $X_O = X_O(t)$ for $0 \le t \le 10\,\mathrm{s}$ and no rolling friction.

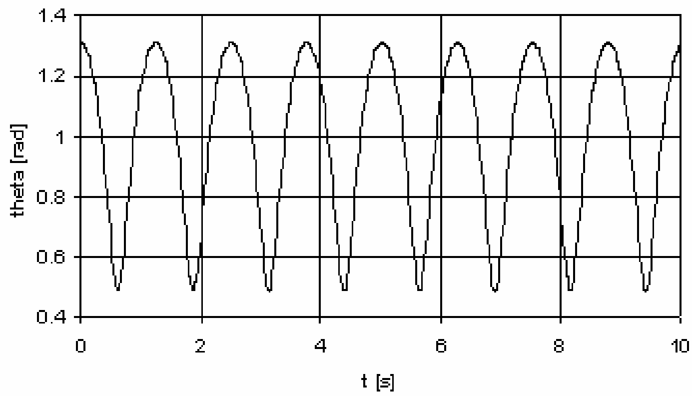

Fig. 2. The variation $\theta = \theta(t)$ for $0 \le t \le 10\,\mathrm{s}$ and no rolling friction.

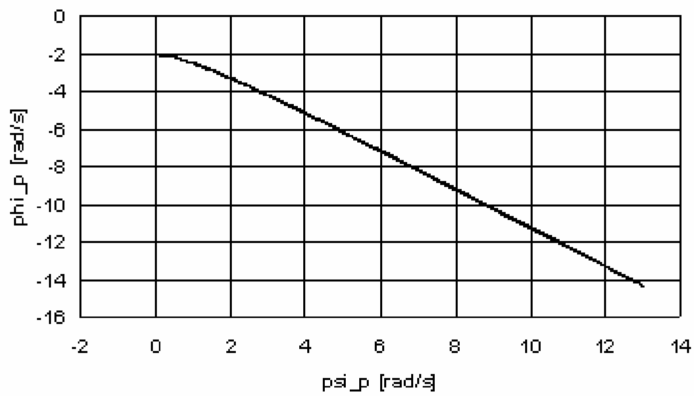

Fig. 3. The variation $\dot\varphi = \dot\varphi(\dot\psi)$ for $0 \le t \le 10\,\mathrm{s}$ and no rolling friction.

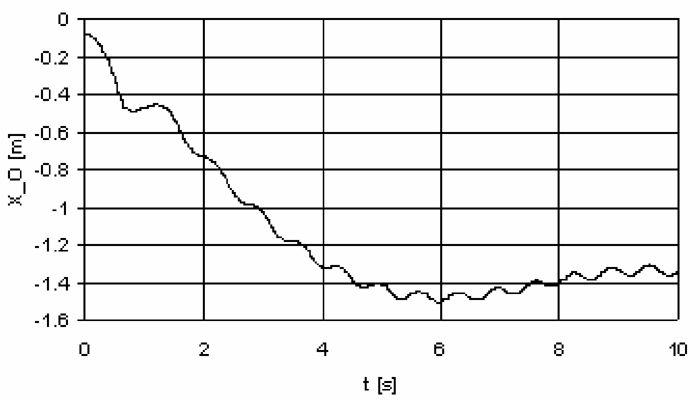

Fig. 4. The variation $X_O = X_O(t)$ for $0 \le t \le 10\,\mathrm{s}$ and with rolling friction.

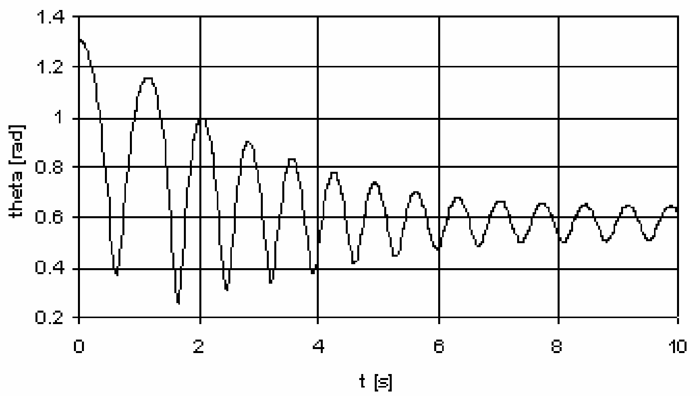

Fig. 5. The variation $\theta = \theta(t)$ for $0 \le t \le 10\,\mathrm{s}$ and with rolling friction.

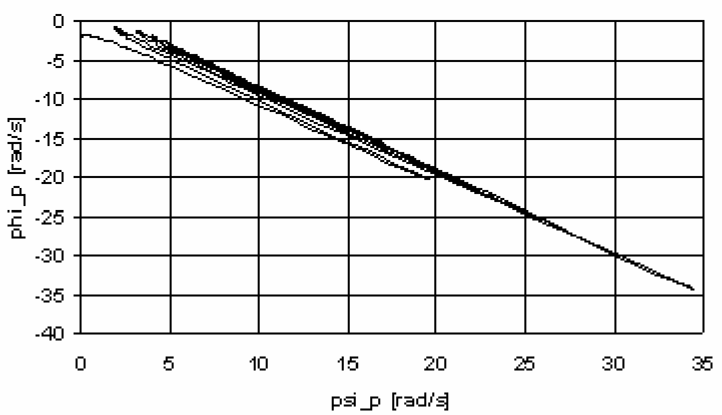

Fig. 6. The variation $\dot{\varphi} = \dot{\varphi}(\dot{\psi})$ for $0 \le t \le 10\,\mathrm{s}$ and with rolling friction.

CONCLUSIONS

In this paper we discussed the free torus wheel acted by its own weight, which is the real case of an automobile envelope wheel. The problem is solved using a multibody type method. It is easy to observe that the equations of motion are more complicate in this situation, but they are more accurate to the practical case. Two particular cases were treated in our paper: with and without rolling friction. From the diagrams presented in the figures, it results that in the case of no rolling friction the motion has a quasi-periodic aspect, but the rolling friction diminishes the amplitude of the motion and it also cancels the quasi-periodic aspect of the motion. In the future we shall discuss the axis of an automobile.

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