AN EQUIVALENT NON-LINEARIZATION METHOD FOR ANALYZING RESPONSE OF NONLINEAR SYSTEMS TO RANDOM EXCITATION

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ABSTRACT-In this article, we are analyzing a differential equation with random variable. Our new technique based on the combination of the transformation method with equivalent linearization theory and minimizing the expectation of the square error to evaluate the probability density function and the power spectral density of the solution. The accuracy of the procedure depends on the bandwidth of the excitation and of the way to decompose the nonlinear restoring force in one linear component plus a nonlinear component. Exact solutions for a non-linear system under random excitation are rare. It is known that even under ideal white noise excitation, only for certain types of non-linear systems, the exact probability density function of the response in the steady state can be obtained [1]. Usually the power spectral density of the input is non-white and the probability density function is taken to be Gaussian to seek an approximate solution through equivalent linearization techniques [2].

1. SYSTEM MODEL

Consider the following oscillator with a nonlinear restoring force component. The ordinary differential equation of the motion can be written as:

$$m\ddot{\eta}(t) + c\dot{\eta}(t) + g(\eta(t)) = F(t), \tag{1}$$

where m is the mass, c is the viscous damping coefficient, $F(t)$ is the external excitation signal with zero mean and $\eta(t)$ is the displacement response of the system. Assuming stationary Gaussian white noise excitation, statistics of the stationary response can be obtained using the Fokker-Planck equation [3].
Dividing the equation by m , the equation of motion can be rewritten as:

$$\ddot{\eta}(t) + 2\xi p\dot{\eta}(t) + h(\eta(t)) = f(t), \tag{2}$$

where f(t) is a zero mean stationary Gaussian white noise excitation, i.e. a power spectral density $S'_0 = \frac{S_F}{m^2} = 1$.
We can always find a way to decompose the nonlinear restoring force to one linear component plus a nonlinear component

$$h(\eta) = p^2(\eta + \frac{1}{\beta}G(\eta)), \tag{3}$$

where β is the nonlinear factor to control the type and degree of nonlinearity in the system. We consider in this article the nonlinear factor $G(\eta)$ of hyperbolical form $G(\eta(t))=sh\left(\beta\eta(t)\right)$. The equation of motion, in this case, can be rewritten as:

$$\ddot{\eta}\left(t\right)+2\xi p\dot{\eta}\left(t\right)+p^2\eta(t)+\frac{1}{\beta}p^2 sh\left(\beta\eta(t)\right)=f\left(t\right), \tag{4}$$

where f(t) is a zero mean stationary Gaussian white noise excitation, with the power spectral density $S'_0=\frac{S_F}{m^2}=1, \xi=\frac{c}{2pm}, sh(\beta\eta(t)=\frac{e^{\beta\eta(t)}-e^{-\beta\eta(t)}}{2}$.

Obtain

$$\ddot{\eta}(t)+2\xi_e p_e\dot{\eta}(t)+p_e{}^2\eta(t)=w(t), \tag{5}$$

where p_e is the undamped natural frequency and ξ_e is the critical damping factor. For the linear system

$$\xi_e=\frac{p}{p_e}\xi. \tag{6}$$

The difference between the nonlinear stiffness and linear stiffness terms is

$$e=p^2[\eta(t)+\frac{1}{\beta}sh\left(\beta\eta(t)\right)]-p^2{}_e\eta(t). \tag{7}$$

The value of p_e can be obtained by minimizing the expectation of the square error

$$\frac{dE\{e^2\}}{dp_e{}^2}=0. \tag{8}$$

Because

$$E\{\eta^2(t)\}=\sigma_\eta{}^2=\int_{-\infty}^{\infty}\eta^2(t)P(\eta(t))d\eta, \tag{9}$$

obtain for p_e

$$p_e{}^2=p^2\left(1+\alpha\frac{E\{\eta(t)\frac{e^{\beta\eta(t)}-e^{-\beta\eta(t)}}{2}\}}{\sigma_\eta{}^2}\right). \tag{10}$$

The forward Fokker Planck equation [6, 9] which governs the transitional probability density function P of system (4) is obtained as follows

$$\frac{\partial P}{\partial t}+\dot{\eta}\frac{\partial P}{\partial\eta}=\frac{\partial}{\partial\dot{\eta}}\left[2\xi p\dot{\eta}P+p^2[\eta(t)+\frac{1}{\beta}sh\left(\beta\eta(t)\right)]P\right]+\pi S'_0\frac{\partial^2 P}{\partial\dot{\eta}^2}. \tag{11}$$

We can write

$$\frac{\partial}{\partial\dot{\eta}}\left[p^2[\eta(t)+\frac{1}{\beta}sh\left(\beta\eta(t)\right)]P+\frac{\pi S'_0}{2\xi p}\frac{\partial P}{\partial\eta}\right]+\left(2\xi p\frac{\partial}{\partial\dot{\eta}}-\frac{\partial}{\partial\eta}\right)\left[\dot{\eta}P+\frac{\pi S'_0}{2\xi p}\frac{\partial P}{\partial\dot{\eta}}\right]=0, \tag{12}$$

from where result

$$p^2[\eta(t)+\frac{1}{\beta}sh\left(\beta\eta(t)\right)]P+\frac{\pi S'_0}{2\xi p}\frac{\partial P}{\partial\eta}=0, \tag{13}$$

$$\dot{\eta}P+\frac{\pi S'_0}{2\xi p}\frac{\partial P}{\partial\dot{\eta}}=0. \tag{14}$$

The density function of the sistem is

$$P(\eta)=C_1 \exp\left(\frac{-2\xi p}{\pi S'_0}\int_0^{\eta} p^2[u+\frac{1}{\beta}sh(\beta u)]du\right), \tag{15}$$

$$P(\dot{\eta})=C_2 \exp\left\{\frac{-2\xi p}{\pi S'_0}\frac{\dot{\eta}^2}{2}\right\}, \tag{16}$$

where C_1 and C_2 are normalisation constants.
We know this

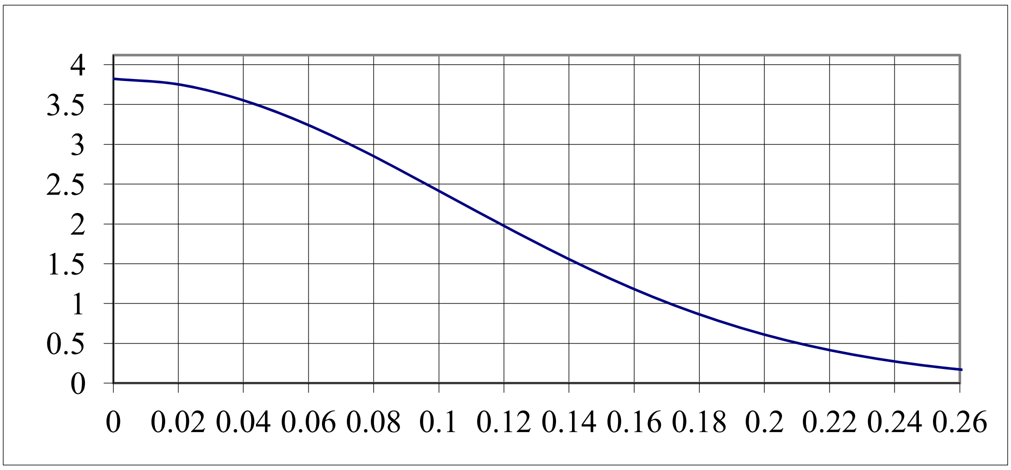
$$\int_{-\infty}^{\infty} e^{-a\eta^2}d\eta=\sqrt{\frac{\pi}{a}}, \int_{-\infty}^{\infty}P(\eta)d\eta=1 . \tag{17}$$

We obtain a solution for the stationary joint probability density function P as

$$P(\eta)=C_1\, \mathrm{e}^{\frac{-2\xi p}{\pi S'_0}\int_0^{\eta}p^2[u+\frac{1}{\beta}sh(\beta u)]du}=C_1 e^{-\frac{2\xi p^3}{\pi S'_0}\eta^2}, \tag{18}$$

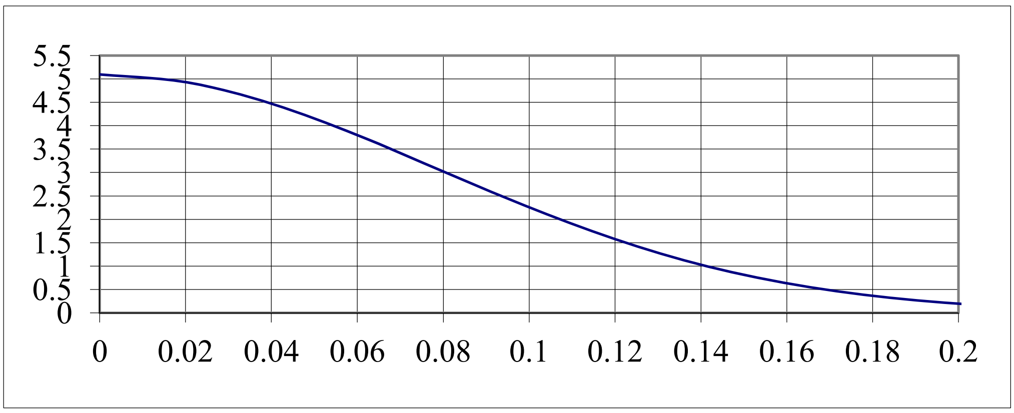
where the constant of normalises is

$$C_1=\frac{p}{\pi}\sqrt{\frac{2\xi p}{S'_0}} . \tag{19}$$



η [m]

Fig. 1. The probability density $P(\eta)\,[m^{-1}]$ for $m=1kg$, $p=6s^{-1}$, $\xi=0,33$, $S_F=1N^2\cdot s$.



η [m]

Fig. 2. The probability density $P(\eta)\,[m^{-1}]$ for $m=1kg$, $p=8s^{-1}$, $\xi=0,08$, $S_F=1N^2\cdot s$.

We calculate the operator $E\{\eta(t)\dfrac{e^{\beta\eta(t)}-e^{-\beta\eta(t)}}{2}\}$. We have

$$E\{\eta(t)\frac{e^{\beta\eta(t)}-e^{-\beta\eta(t)}}{2}\}=\frac{p}{2\pi}\sqrt{\frac{2\xi p}{S_0^{'}}}\left[\int_{-\infty}^{\infty}\eta e^{\beta\eta-\frac{2\xi p^3}{\pi S_0^{'}}\eta^2}d\eta-\int_{-\infty}^{\infty}\eta e^{-\beta\eta-\frac{2\xi p^3}{\pi S_0^{'}}\eta^2}d\eta\right]. \qquad (20)$$

We know this

$$\int_{-\infty}^{\infty}\eta e^{-(a\eta^2+b\eta)}d\eta=-\frac{1}{2}\sqrt{\frac{\pi}{a^3}}be^{\frac{b^2}{4a}} \qquad (21)$$

Obtain

$$E\{\eta(t)\frac{e^{\beta\eta(t)}-e^{-\beta\eta(t)}}{2}\}=\frac{\pi S_0^{'}\beta}{4\xi p^3}e^{\frac{\pi S_0\beta^2}{8\xi p^3}}. \qquad (22)$$

The standard deviation of $\eta(t)$ is

$$\sigma_\eta^2=\int_{-\infty}^{\infty}\eta^2P(\eta)d\eta=\frac{p}{\pi}\sqrt{\frac{2\xi p}{S_0^{'}}}\int_{-\infty}^{\infty}\eta^2e^{-\frac{2\xi p^3}{\pi S_0^{'}}\eta^2}d\eta. \qquad (23)$$

We know this

$$\int_{-\infty}^{\infty}\eta^2e^{-a\eta^2}d\eta=\frac{1}{2}\sqrt{\frac{\pi}{a^3}}. \qquad (24)$$

Obtain

$$\sigma_\eta^2=\frac{\pi S_0^{'}}{4\xi p^3}. \qquad (25)$$

The expression of p_e can be obtained as

$$p_e^{\,2}=p^2\left(1+e^{\frac{\pi S_0^{'}\beta^2}{8\xi p^3}}\right). \qquad (26)$$

Using the Fourier transform of equation [4,5] we obtain

$$\bar{\eta}(\omega)(p_e^{\,2}-\omega^2+2\xi_e p_e\omega i)=\bar{F}(\omega), \qquad (27)$$

where $\mathrm{F}(\dot{\eta}(t))=i\omega\bar{\eta}(\omega)$, $\mathrm{F}(F(t))=\bar{F}(\omega)$.

The frequency response function [4,5] of the system is given by equation

$$\frac{1}{H(\omega)}=k_e^{\,2}-m\omega^2+c\omega i=m(p_e^{\,2}-\omega^2+2\xi p\omega i), \qquad (28)$$

$$\bar{\eta}(\omega)=\bar{F}(\omega)H(\omega). \qquad (29)$$

The power spectral density of response is

$$S_\eta(\omega)=\left|H(\omega)\right|^2S_F(\omega), \qquad (30)$$

or

$$S_\eta(\omega)=\frac{S_0^{'}}{m\left[(p_e^{\,2}-\omega^2)^2+4\xi_e^{\,2}p_e^{\,2}\omega^2\right]}=\frac{S_F}{m^2\left[(p_e^{\,2}-\omega^2)^2+4\xi_e^{\,2}p_e^{\,2}\omega^2\right]}. \qquad (31)$$

We obtain for the power spectral density of response [6,7,8]

$$S_{\eta}(\omega)=\frac{\frac{S_F}{m^2}}{\left(p^2+\frac{p^2E\{\eta G(\eta)\}}{C_1\beta\int_0^{\infty}\eta^2 e^{\frac{-2\xi p}{\pi S'_0}\int_0^{\eta}p^2(u+\frac{1}{\beta}G(u))du}d\eta}-\omega^2\right)^2+4\xi^2p^2\omega^2}=\frac{\frac{S_F}{m^2}}{\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]^2+4\xi^2p^2\omega^2}. \tag{32}$$

The power interspectral density of response [5,6] is given by equation

$$S_{\eta F}(\omega)=H(\omega)S_F(\omega)=\frac{mS'_0}{p_e{}^2-\omega^2+2\xi p\omega i}. \tag{33}$$

We obtain

$$S_{\eta F}(\omega)=\frac{S'_0}{p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2+2\xi p\omega i}. \tag{34}$$

The power interspectral density by the coincidence [8,9] is

$$S_{\eta Fr}(\omega)=\frac{mS'_0\left(p^2+\frac{p^2E\{\eta G(\eta)\}}{C_1\beta\int_0^{\infty}\eta^2 e^{\frac{-2\xi p}{\pi S'_0}\int_0^{\eta}h(u)du}d\eta}-\omega^2\right)}{\left(p^2+\frac{p^2E\{\eta G(\eta)\}}{C_1\beta\int_0^{\infty}\eta^2 e^{\frac{-2\xi p}{\pi S'_0}\int_0^{\eta}h(u)du}d\eta}-\omega^2\right)^2+4\xi^2p^2\omega^2}=\frac{mS'_0\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]}{\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]^2+4\xi^2p^2\omega^2}. \tag{35}$$

The power interspectral density by the quadrature [8,9] is

$$S_{\eta Fc}(\omega)=\frac{-2\xi p\omega mS'_0}{\left(p^2+\frac{p^2E\{\eta G(\eta)\}}{C_1\beta\int_0^{\infty}\eta^2 e^{\frac{-2\xi p}{\pi S'_0}\int_0^{\eta}h(u)du}d\eta}-\omega^2\right)^2+4\xi^2p^2\omega^2}=\frac{-2\xi p\omega mS'_0}{\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]^2+4\xi^2p^2\omega^2}. \tag{36}$$

2. THE NUMERICAL RESULTS

For $m=1kg$, $k=36\frac{N}{m}$, $c=4\frac{Ns}{m}$, $\beta=7s^{-1}$, with $S_F=1N^2\cdot s$, we will find the statistical parameters of function. We obtain

$$p=\sqrt{\frac{k}{m}}=6s^{-1},\ \frac{c}{m}=2\xi p \Rightarrow \xi=0,33, \tag{37}$$

$$\sigma_\eta^2=\frac{\pi S'_0}{4\xi p^3}=0,0109m^2.(\text{for } S'_0=1\frac{m^2}{s^3}), \tag{38}$$

$$p_e^{\,2}=p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)=82,8s^{-2} \Rightarrow p_e=9,09s^{-1}. \tag{39}$$

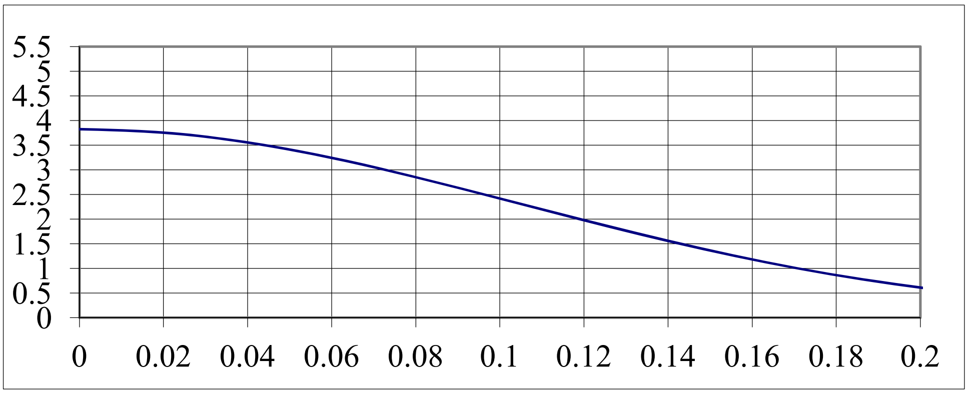
The interspectral density of response is

$$S_{\eta F}(\omega)=\frac{mS'_0}{p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2+2\xi p\omega i}=\frac{1}{82,8-\omega^2+3,96\omega i}. \tag{40}$$

The power interspectral density by the coincidence respectively the power interspectral density by the quadrature are given by the equations

$$S_{\eta fr}(\omega)=\frac{p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2}{\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]^2+4\xi^2p^2\omega^2}mS'_0=\frac{82,8-\omega^2}{(82,8-\omega^2)^2+15,68\omega^2}, \tag{41}$$

$$S_{\eta fc}(\omega)=-\frac{2\xi p\omega}{\left[p^2\left(1+e^{\frac{\pi S'_0\beta^2}{8\xi p^3}}\right)-\omega^2\right]^2+4\xi^2p^2\omega^2}=-\frac{3,96\omega}{(82,8-\omega^2)^2+15,68\omega^2}mS'_0. \tag{42}$$



η [m]

Fig.3. The probability density $P(\eta)\,[m^{-1}]$ for $p=6s^{-1}, \xi=0,33$.

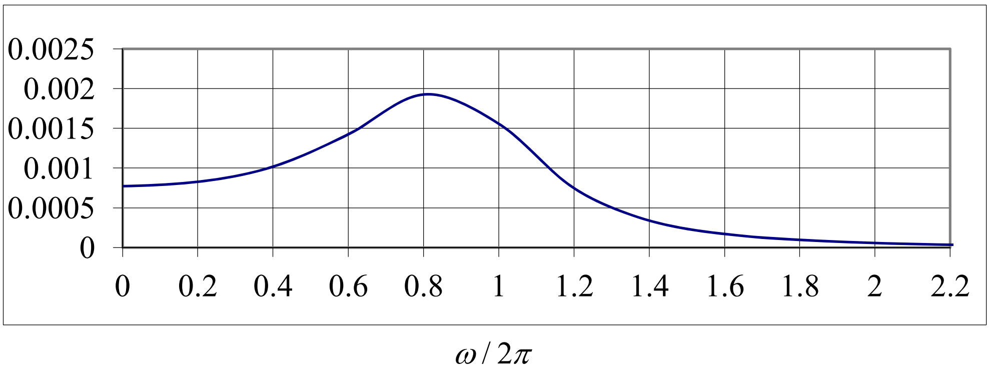


Fig. 4. The power spectral density $S_\eta\ [m^2 \cdot s]$ for $m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \beta = 7m^{-1}$.

3. CONCLUSIONS

The stationary probability densities obtained for Duffing van der Pol oscillator under both external and parametric random excitations by using the linearization procedure are well verified by the results from simulation of original equation of motion. In this context, a system widely studied has been the nonlinear oscillators under white noise excitation. The availability of an exact solution for this system under white noise excitation helps one to understand the limitations of approximate methods. Apart from results on the stationary probability density function, one would be interested in the response power spectral density function. In linear problems, the power spectral density is easily found as the product of the input power spectral density and the system frequency response function, for any arbitrary input probability density function. Such a facility is not available with the simplest non-linear system. This method is restricted to wideband excitations and lightly damped systems, such that the response can be taken to be a narrow band process. No general method is available at present to obtain the response probability density function and the power spectral density of a non-linear system under a given arbitrary Gaussian random input. Detailed numerical results are presented for of nonlinear oscillators under white noise excitation. The exact probability structure of this special input is found. However, the condition derived is only sufficient, and hence the solution presented is not unique.

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