THE SPECTRAL RESPONSE FOR THE RANDOM EXCITATION OF THE ELASTIC PENDULUM

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ABSTRACT - We present a method for estimating the power spectral density of the stationary response of the simple pendulum consist of a mass m, approximated here as point-mass which is attached to the lower end of light, rigid rod, of length l. The upper end of the rod is free to pivot about point, at 0. If attention is restricted to oscillation of the pendulum in a vertical plane the one has a single degree of freedom system, approximately. Numerous applications of this technique for studying the response of non-linear oscillators to random excitation have described in the literature (Roberts, Spanos). An equivalent linear system is derived, from which the power spectral density is deduced. The theoretical analyses are verified by numerical results.

1. SYSTEM MODEL

The pendul from the below figure, formed from a lenght bar l with insignificant mass, caught in point O with the spring with the k elasticity constant on which hangs the body with the mass.

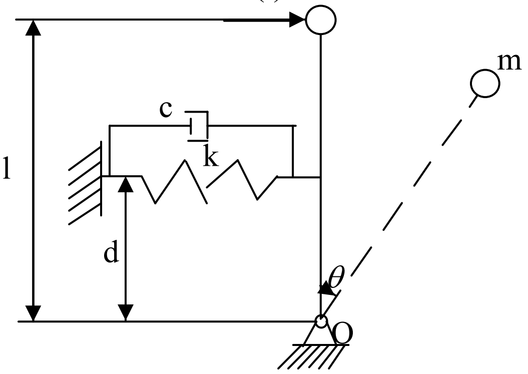


Fig 1. The elastic pendulum.

The ordinary differential equation of the motion can be written as:

$$J_0\ddot{\theta} = mgl\sin\theta - kd^2\sin\theta\cos\theta - cd^2\cos^2\theta\cdot\dot{\theta} + F(t)l\ , \tag{1}$$

or

$$ml^2\ddot{\theta} + kd^2\sin\theta\cos\theta + cd^2\cos^2\theta\cdot\dot{\theta} - mgl\sin\theta = F(t)l\ , \tag{2}$$

or

$$\ddot{\theta} + \frac{cd^2}{ml^2}\cos^2\theta\cdot\dot{\theta} + \frac{kd^2}{ml^2}\sin\theta\cos\theta - \frac{g}{l}\sin\theta = \frac{F(t)}{ml}\ . \tag{3}$$

If we consider the Tylor developments around point 0 for the function $\sin\theta$ and $\cos\theta$, form where we keep just the first two termens, we can write

$$\sin\theta = \theta - \frac{\theta^3}{6}, \cos\theta = 1 - \frac{\theta^2}{2} \quad (4)$$

and equation of motion, while neglecting very small terms we get

$$\ddot{\theta} + \frac{cd^2}{ml^2}(1-\lambda\theta^2)\dot{\theta} + \left(\frac{kd^2}{ml^2} - \frac{g}{l}\right)\theta + \alpha\left(\frac{g}{6l} - \frac{2kd}{3ml^2}\right)\theta^3 = u(t), \quad (5)$$

where $u(t) = \frac{F(t)}{ml}$, λ and α are the nonlinear factors to control the type and degree of nonlinearity in the system [1].

If we notate

$$h(\theta(t),\dot{\theta}(t)) = \frac{cd^2}{ml^2}(1-\lambda\theta^2)\dot{\theta} + \left(\frac{kd^2}{ml^2} - \frac{g}{l}\right)\theta + \alpha\left(\frac{g}{6l} - \frac{2kd}{3ml^2}\right)\theta^3, \quad (6)$$

the equation of motion is

$$\ddot{\theta}(t) + \beta_{ech}\dot{\theta}(t) + \gamma_{ech}\theta(t) = w(t). \quad (7)$$

The difference between the nonlinear stiffness [2,3] and linear stiffness terms is

$$\varepsilon = h(\theta(t),\dot{\theta}(t)) - \beta_{ech}\dot{\theta}(t) - \gamma_{ech}\theta(t). \quad (8)$$

The value of p_e can be obtained by minimizing the expectation of the square error

$$\frac{\partial}{\partial \beta_{ech}} E[\varepsilon^2] = 0, \quad (9)$$

$$\frac{\partial}{\partial \gamma_{ech}} E[\varepsilon^2] = 0. \quad (10)$$

Because [6,8]

$$E\{\varepsilon^2\} = E\{h^2\} + \beta_{ech}^2 E\{\dot{\eta}^2\} + \gamma_{ech}^2 E\{\eta^2\} - 2\beta_{ech}E\{\dot{\theta}h\} + 2\beta_{ech}\gamma_{ech}E\{\dot{\theta}\theta\} - 2\gamma_{ech}E\{\theta h\}, \quad (11)$$

we have

$$E\{\dot{\theta}h\} - \beta_{ech}E\{\dot{\theta}^2\} - \gamma_{ech}E\{\theta\dot{\theta}\} = 0, \quad (12)$$

$$E\{\theta h\} - \beta_{ech}E\{\theta\dot{\theta}\} - \gamma_{ech}E\{\theta^2\} = 0. \quad (13)$$

Obtain the solutions

$$\beta_{ech} = \frac{E\{\theta^2\}E\{\dot{\theta}h\} - E\{\theta\dot{\theta}\}E\{\theta h\}}{E\{\theta^2\}E\{\dot{\theta}^2\} - (E\{\theta\dot{\theta}\})^2}, \quad (14)$$

$$\gamma_{ech} = \frac{E\{\dot{\theta}^2\}E\{\theta h\} - E\{\theta\dot{\theta}\}E\{\dot{\theta}h\}}{E\{\theta^2\}E\{\dot{\theta}^2\} - (E\{\theta\dot{\theta}\})^2}. \quad (15)$$

Because $E\{\theta\dot{\theta}\} = 0$, the solutions are simplified

$$\beta_{ech} = \frac{E\{\dot{\theta}h\}}{E\{\dot{\theta}^2\}}, \quad \gamma_{ech} = \frac{E\{\eta h\}}{E\{\eta^2\}}. \quad (16)$$

Obtain the linearization equation [2,3,5,6]

$$\ddot{\theta}(t) + \frac{E\{\dot{\theta}h\}}{E\{\dot{\theta}^2\}}\dot{\theta}(t) + \frac{E\{\theta h\}}{E\{\theta^2\}}\theta(t) = u(t), \quad (17)$$

where

$$\frac{E\{\dot{\theta}h\}}{E\{\dot{\theta}^2\}}=\frac{cd^2}{ml^2}\left[\frac{E\{\dot{\theta}^2\}}{E\{\dot{\theta}^2\}}-\lambda\frac{E\{\dot{\theta}^2\theta^2\}}{E\{\dot{\theta}^2\}}\right]+\left(\frac{kd^2}{ml^2}-\frac{g}{l}\right)\frac{E\{\dot{\theta}\theta\}}{E\{\dot{\theta}^2\}}+\alpha\left(\frac{g}{6l}-\frac{2kd^2}{3ml^2}\right)\frac{E\{\dot{\theta}\theta^3\}}{E\{\dot{\theta}^2\}}, \quad (18)$$

$$\frac{E\{\theta h\}}{E\{\theta^2\}}=\frac{cd^2}{ml^2}\left(\frac{E\{\theta\dot{\theta}\}}{E\{\theta^2\}}-\lambda\frac{E\{\theta^3\dot{\theta}\}}{E\{\theta^2\}}\right)+\left(\frac{kd^2}{ml^2}-\frac{g}{l}\right)\frac{E\{\theta^2\}}{E\{\theta^2\}}+\alpha\left(\frac{g}{6l}-\frac{2kd^2}{3ml^2}\right)\frac{E\{\theta^4\}}{E\{\theta^2\}}. \quad (19)$$

Because

$$E\{\dot{\theta}\theta\}=0\,;E\{\dot{\theta}\theta^3\}=0\,;E\{\theta^2\}=\sigma^2_{\theta}\,;E\{\theta^4\}=3\sigma^4_{\theta}\,;\frac{E\{\dot{\theta}^2\theta^2\}}{E\{\dot{\theta}^2\}}=2\sigma^2_{\theta}, \quad (20)$$

the relations (18) and (19) become

$$\frac{E\{\dot{\theta}h\}}{E\{\dot{\theta}^2\}}=\frac{cd^2}{ml^2}\left(1-2\lambda\sigma^2_{\theta}\right), \quad (21)$$

$$\frac{E\{\theta h\}}{E\{\theta^2\}}=\left(\frac{kd^2}{ml^2}-\frac{g}{l}\right)+3\alpha\left(\frac{g}{6l}-\frac{2kd^2}{3ml^2}\right)\sigma^2_{\theta}. \quad (22)$$

The equation of motion can be write

$$\ddot{\theta}+\frac{cd^2}{ml^2}\left(1-2\lambda\sigma^2_{\theta}\right)\dot{\theta}+\left[\left(\frac{kd^2}{ml^2}-\frac{g}{l}\right)+3\alpha\left(\frac{g}{6l}-\frac{2kd^2}{3ml^2}\right)\sigma^2_{\theta}\right]\theta=u(t). \quad (23)$$

Using the Fourier transform [5,8,9] of equation (23.) we obtain for the frequency response function

$$H(\omega)=\frac{1}{-m\omega^2+i\omega\frac{cd^2}{l^2}\left(1-2\lambda\sigma^2_{\theta}\right)+\left(\frac{kd^2}{l^2}-\frac{mg}{l}\right)+3\alpha\left(\frac{mg}{6l}-\frac{2kd^2}{3l^2}\right)\sigma^2_{\theta}}. \quad (24)$$

The power interspectral density of response [3] is given by equation

$$S_{\theta}(\omega)=\frac{S_F(\omega)}{\left[\frac{kd^2}{l^2}-\frac{mg}{l}+3\alpha\left(\frac{mg}{6l}-\frac{2kd^2}{3l^2}\right)\sigma^2_{\theta}-m\omega^2\right]^2+\frac{c^2d^4}{l^4}\left(1-2\lambda\sigma^2_{\theta}\right)^2\omega^2}. \quad (25)$$

The mean square value for the displacement of the system is

$$\sigma^2_{\theta}=\int_{-\infty}^{\infty}S_{\theta}(\omega)d\omega, \quad (26)$$

or

$$\sigma^2_{\theta}=\frac{\pi S_F}{\frac{cd^2}{l^2}\left(1-2\lambda\sigma^2_{\theta}\right)\left[\left(\frac{kd^2}{l^2}-\frac{mg}{l}\right)+3\alpha\left(\frac{mg}{6l}-\frac{2kd^2}{3l^2}\right)\sigma^2_{\theta}\right]}. \quad (27)$$

The velocity variance of the system is

$$\sigma^2_{\dot{\theta}}=\frac{\pi S_F l^2}{mcd^2\left(1-2\lambda\sigma^2_{\theta}\right)}. \quad (28)$$

The value of 0 p_e is

$$p_e = \sqrt{\left(\frac{kd^2}{ml^2} - \frac{g}{l}\right) + 3\alpha\left(\frac{g}{6l} - \frac{2kd^2}{3ml^2}\right)\sigma^2_\theta} \; . \tag{29}$$

2. NUMERICAL RESULTS

The pendul formed from a lenght bar $l = 0,5m$, with insignificant mass, caught in point O,with the elasticity constant $k = 36\frac{N}{m}$, in which hangs the body with the $m = 1kg$ mass and $c = 1\frac{N \cdot s}{m}, d = 0,4m, \alpha = 5rad^{-2}, \lambda = 0,34rad^{-2}$.

We consider the power spectral density $S_F = 1N^2 \cdot s$.

Obtain

$$\sigma^2_\theta = \frac{\pi}{0,5 \cdot 1,28\left(1-0,68\sigma^2_\theta\right)\left(26,48-13,5\sigma^2_\theta\right)}, \tag{30}$$

or

$$\sigma^2_\theta = 0,26rad^2, \tag{31}$$

where

$$\sigma_\theta = 29^0 10'. \tag{32}$$

Obtain

$$\sigma^2_{\dot{\theta}} = \frac{\pi S_F l^2}{mcd^2\left(1-2\lambda\sigma^2_\theta\right)} = 0,58\frac{rad^2}{s^2}, \tag{33}$$

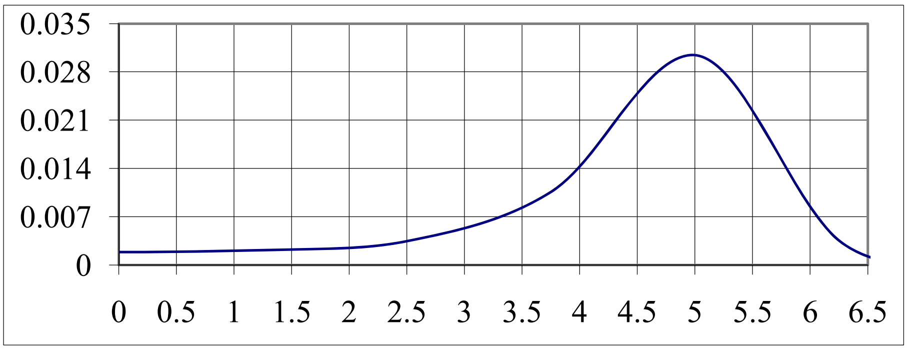
$$\sigma_{\dot{\theta}} = 0,76\frac{rad}{s}. \tag{34}$$

The undamped natural frequency is

$$p_e = \sqrt{\left(\frac{kd^2}{ml^2} - \frac{g}{l}\right) + 3\alpha\left(\frac{g}{6l} - \frac{2kd^2}{3ml^2}\right)\sigma^2_\theta} = 4,79s^{-1}. \tag{35}$$

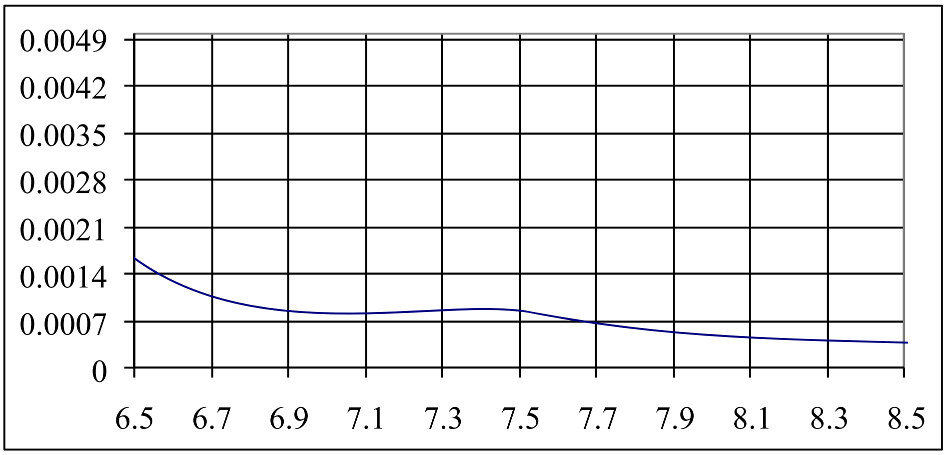
The power spectral density of response is

$$S_\theta(\omega) = \frac{S_F}{\left[22,97 - m\omega^2\right]^2 + 1,1\omega^2}. \tag{36}$$



$$\frac{\omega}{2\pi}$$

Fig 2.The diagrame of the power spectral density of response $S_\theta(\omega)$ $[rad^2 \cdot s]$



$\frac{\omega}{2\pi}$

Fig 3. The diagrame of the power spectral density of response $S_{\theta}(\omega)$ $[rad^2 \cdot s]$

3. REMARKS

We observ that the maximum value of 0,03 $rad^2 \cdot s$ are obtained for values of the frequency of 5 Hz although , in the frequency band of values untill 2 Hz the spectral power density is almost costant 0,0021 $rad^2 \cdot s$. The fast lowering in the interval $(5; 6,25)$ Hz.as you can see in the graf below in the frequency band $(6,5; 8,5)$ in which the spectral density is low untill 0,0004 $rad^2 \cdot s$ we can see the density is constant for 0,00082 $rad^2 \cdot s$ for frequencies in the interval $(7,1; 7,5)$ Hz.

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