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THE RIGID BODY WITH A FIXED POINT IN THE LAGRANGE–POISSON CASE WITH APPLICATION TO KERS DYNAMICS

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Abstract: *In our paper we discuss the dynamics of the rigid body with a fixed point in the Lagrange–Poisson case, obtaining the condition for the regular precession. This condition is applied to a particular KERS (Kinetic Energy Recovery System) resulting the condition that must exist between different geometric parameters for a regular precession. A numerical application is also developed.*

Keywords: Lagrange–Poisson, KERS, regular precession.

THE LAGRANGE–POISSON CASE

We consider the rigid body that has the fixed point at O (Fig. 1) and the fixed reference system and the mobile reference system rigidly linked to the solid rigid, respectively, $OXYZ$ and $Oxyz$, respectively.

The OZ axis is vertical ascendant, and the mobile axes are principal inertial axes linked to the rigid body at the point O; hence we may write:

$$J_{xy} = J_{xz} = J_{yz} = 0. \tag{1}$$

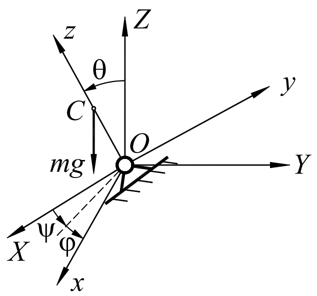

Figure 1. Rigid body with a fixed point in the case Lagrange–Poisson.

For this case is also known that the inertial ellipsoid is a rotational one about the Oz axis, that is

$$J_x = J_y. \tag{2}$$

In addition, the center of weight C is situated on the Oz axis and we denote

$$OC = \zeta. \tag{3}$$

MATRIX OF ROTATION. MATRIX ANGULAR VELOCITY

Denoting by $[\psi]$, $[\theta]$, $[\varphi]$ the matrices given by

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$$[\boldsymbol{\psi}] = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},\ [\boldsymbol{\theta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},\ [\boldsymbol{\varphi}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

it results the rotational matrix

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}] = \begin{bmatrix} \mathrm{c}\,\psi\,\mathrm{c}\,\varphi - \mathrm{s}\,\psi\,\mathrm{c}\,\theta\,\mathrm{s}\,\varphi & -\mathrm{c}\,\psi\,\mathrm{s}\,\varphi - \mathrm{s}\,\psi\,\mathrm{c}\,\theta\,\mathrm{c}\,\varphi & \mathrm{s}\,\psi\,\mathrm{s}\,\theta \\ \mathrm{s}\,\psi\,\mathrm{c}\,\varphi + \mathrm{c}\,\psi\,\mathrm{c}\,\theta\,\mathrm{s}\,\varphi & -\mathrm{s}\,\psi\,\mathrm{s}\,\varphi + \mathrm{c}\,\psi\,\mathrm{c}\,\theta\,\mathrm{c}\,\varphi & -\mathrm{c}\,\psi\,\mathrm{s}\,\theta \\ \mathrm{s}\,\theta\,\mathrm{s}\,\varphi & \mathrm{s}\,\theta\,\mathrm{c}\,\varphi & \mathrm{c}\,\theta \end{bmatrix}. \tag{5}$$

The matrix $\{\boldsymbol{\omega}\}$ of the angular velocities has the expression

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix}, \tag{6}$$

where

$$[\mathbf{Q}] = [\boldsymbol{\varphi}]^{\mathrm{T}}\left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_{\psi}\}\ \{\mathbf{u}_{\theta}\}\ \{\mathbf{u}_{\varphi}\}\right], \tag{7}$$

$$\{\mathbf{u}_{\psi}\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \{\mathbf{u}_{\theta}\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \{\mathbf{u}_{\varphi}\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

It results

$$[\mathbf{Q}] = \begin{bmatrix} \sin\theta\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & -\sin\varphi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}, \tag{9}$$

$$\{\boldsymbol{\omega}\} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \dot{\psi}\cos\theta + \dot{\varphi} \end{bmatrix}. \tag{10}$$

DETERMINATION OF THE PRIME INTEGRALS

The theorem of the kinetic energy and work offers

$$\mathrm{d}T = \mathrm{d}W, \tag{11}$$

where

$$T = \frac{1}{2}J_x\omega_x^2 + \frac{1}{2}J_x\omega_y^2 + \frac{1}{2}J_z\omega_z^2, \tag{12}$$

while $\mathrm{d}W$ is given by

$$\mathrm{d}W = m\mathbf{g}\cdot\mathrm{d}\mathbf{r}_C. \tag{13}$$

Since

$$\mathbf{g} = -g\mathbf{k}_1,\ \mathbf{r}_C = \zeta\mathbf{k}, \tag{14}$$

where $\mathbf{k}_1$ and $\mathbf{k}$ are the unit vectors of the axes OZ and Oz, respectively, one deduces the expressions

$$\mathbf{k}_1\cdot\mathbf{k} = \cos\theta, \tag{15}$$

$$\mathbf{g}\cdot\mathrm{d}\mathbf{r}_C = -g\zeta\cos\theta. \tag{16}$$

We obtained the relation

$$\mathrm{d}\left(\frac{1}{2}J_x\omega_x^2 + \frac{1}{2}J_x\omega_y^2 + \frac{1}{2}J_z\omega_z^2\right) = -mg\zeta\mathrm{d}(\cos\theta)\mathrm{t} \tag{17}$$

and, by integration, we get

$$\frac{1}{2}J_x\omega_x^2 + \frac{1}{2}J_x\omega_y^2 + \frac{1}{2}J_z\omega_z^2 + mg\zeta\cos\theta = C_1, \tag{18}$$

where C_1 is a constant of integration.

Theorem of the moment of momentum relative to the OZ axis gives

$$\dot{K}_Z = M_Z = 0 . \tag{19}$$

But

$$\{\mathbf{K}_O\} = [\mathbf{J}]\{\boldsymbol{\omega}\}, \tag{20}$$

where

$$\{\mathbf{K}_O\} = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix}, \tag{21}$$

$$[\mathbf{J}] = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_x & 0 \\ 0 & 0 & J_z \end{bmatrix}; \tag{22}$$

it follows:

$$\mathbf{K}_O = J_x\omega_x\mathbf{i} + J_x\omega_y\mathbf{j} + J_z\omega_z\mathbf{k} , \tag{23}$$

$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ being the unit vectors of the mobile axes.

From the relation

$$\{\mathbf{X}\} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [\mathbf{A}]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\mathbf{A}]\{\mathbf{x}\} \tag{24}$$

that links the coordinates in the two systems, we find

$$\{\mathbf{x}\} = [\mathbf{A}]^{\mathrm{T}}\{\mathbf{X}\}; \tag{25}$$

for the point of coordinates $[0\ \ 0\ \ 1]^{\mathrm{T}}$ relative to the fixed system, the last relation becomes:

$$\{\mathbf{x}\} = [\mathbf{A}]^{\mathrm{T}}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ \cos\theta \end{bmatrix}, \tag{26}$$

that is

$$\mathbf{k}_1 = \sin\theta\sin\varphi\mathbf{i} + \sin\theta\cos\varphi\mathbf{j} + \cos\theta\mathbf{k} . \tag{27}$$

On the other hand

$$K_Z = \mathbf{K}_O \cdot \mathbf{k}_1 , \tag{28}$$

so that we have

$$K_z = J_x\omega_x \sin\theta\sin\varphi + J_x\omega_y \sin\theta\cos\varphi + J_z\omega_z \cos\theta . \tag{29}$$

The expression (19) leads to the prime integral of the moment of momentum:

$$J_x\omega_x \sin\theta\sin\varphi + J_x\omega_y \sin\theta\cos\varphi + J_z\omega_z \cos\theta = C_2 , \tag{30}$$

where C_2 is a constant of integration.

EQUATIONS OF MOTION

The Euler equations

$$\begin{cases} J_x\dot{\omega}_x + \omega_y\omega_z\left(J_z - J_y\right) = M_x, \\ J_y\dot{\omega}_y + \omega_z\omega_x\left(J_x - J_z\right) = M_y, \\ J_z\dot{\omega}_z + \omega_x\omega_y\left(J_y - J_x\right) = M_z \end{cases} \tag{31}$$

become now

$$\begin{cases} J_x\dot{\omega}_x + \omega_y\omega_z\left(J_z - J_x\right) = M_x, \\ J_x\dot{\omega}_y + \omega_z\omega_x\left(J_x - J_z\right) = M_y, \\ J_z\dot{\omega}_z = 0. \end{cases} \tag{32}$$

The last relation (32) gives immediately

$$\omega_z = \omega_0 = \text{const}\,. \tag{33}$$

We replace the system (32) by an equivalent system given by the equations (33), (18) and (30), i.e.

$$\begin{cases} \omega_z = \omega_0, \\ J_x\left(\omega_x^2 + \omega_y^2\right) + J_z\omega_z^2 = 2C_1 - 2mg\zeta\cos\theta, \\ J_x\left(\omega_x\sin\varphi + \omega_y\cos\varphi\right)\sin\theta = C_2 - J_z\omega_z\cos\theta \end{cases} \tag{34}$$

or, in an equivalent form,

$$\begin{cases} \omega_z = \omega_0, \\ \omega_x^2 + \omega_y^2 = \dfrac{2C_1}{J_x} - \dfrac{J_z}{J_x}\omega_z^2 - \dfrac{2mg\zeta\cos\theta}{J_x}, \\ \left(\omega_x\sin\varphi + \omega_y\cos\varphi\right)\sin\theta = \dfrac{C_2}{J_x} - \dfrac{J_z}{J_x}\omega_z\cos\theta. \end{cases} \tag{35}$$

Keeping into account the expressions (10), the last system reads

$$\begin{cases} \omega_z = \omega_0, \\ \dot{\psi}^2\sin^2\theta + \dot{\theta}^2 = \dfrac{2C_1 - J_z\omega_0^2}{J_x} - \dfrac{2mg\zeta\cos\theta}{J_x}, \\ \dot{\psi}\sin^2\theta = \dfrac{C_2}{J_x} - \dfrac{J_z}{J_x}\omega_0\cos\theta \end{cases} \tag{36}$$

and denoting

$$D_1 = \frac{2C_1 - J_z\omega_0^2}{J_x},\ D_2 = \frac{C_2}{J_x},\ A_1 = \frac{2mg}{J_x},\ A_2 = \frac{J_z}{J_x}\omega_0, \tag{37}$$

it takes the form

$$\begin{cases} \omega_z = \omega_0, \\ \dot{\psi}^2\sin^2\theta = D_1 - A_1\cos\theta - \dot{\theta}^2, \\ \dot{\psi}\sin^2\theta = D_2 - A_2\cos\theta. \end{cases} \tag{38}$$

We multiply the second relation (38) by $\sin^2\theta$, square the last relation (38), and equate the results

$$D_1\sin^2\theta - A_1\cos\theta\sin^2\theta - \dot{\theta}^2\sin^2\theta = D_2^2 + A_2^2\cos^2\theta - 2D_2A_2\cos\theta, \tag{39}$$

where from

$$\dot{\theta}^2\sin^2\theta = D_1\left(1 - \cos^2\theta\right) - A_1\cos\theta\left(1 - \cos^2\theta\right) - D_2^2 - A_2^2\cos^2\theta + 2D_2A_2\cos\theta, \tag{40}$$

$$\dot{\theta}^2\sin^2\theta = A_1\cos^3\theta - \left(D_1 + A_2^2\right)\cos^2\theta + \left(2D_2A_2 - A_1\right)\cos\theta + D_1 - D_2^2. \tag{41}$$

Denoting

$$w = \cos\theta, \tag{42}$$

we have

$$\dot{w} = \frac{\mathrm{d}w}{\mathrm{d}t} = -\dot{\theta}\sin\theta \tag{43}$$

and the equation (41) becomes

$$\dot{w}^2 = A_1w^3 - \left(D_1 + A_2^2\right)w^2 + \left(2D_2A_2 - A_1\right)w + D_1 - D_2^2. \tag{44}$$

In the right-hand part of the differential equation (44) we have a polynomial of third degree in w, that is, a polynomial that has at least one real root.
Let us write this polynomial in the form

$$f(w) = a_3w^3 + a_2w^2 + a_1w + a_0, \tag{45}$$

with

$$a_3 = A_1 ,\ a_2 = -\left(D_1 + A_2^2\right),\ a_1 = 2D_2A_2 - A_1 ,\ a_0 = D_1 - D_2^2 . \tag{46}$$

REGULAR PRECESSION

The regular precession appears when the polynomial $f(w)$ has a real double root $w_1 = w_2$, $f(w_1) = 0$, $f(w_2) = 0$ and, moreover, this double root coincides to the initial value w_0.
We have

$$f'(w) = 3a_3w^2 + 2a_2w + a_1 ; \tag{47}$$

hence

$$f(w\ \ ) = a_3w^3 + a_2w^2 + a_1w + a_0 ,\ f'(w_1) = 3a_3w_1^2 + 2a_2w_1 + a_1 . \tag{48}$$

Multiplying the first relation (48) by -3, the second relation (48) by w_1 and adding the results, we obtain

$$a_2w_1^2 + 2_1w_1 + 3a_0 = 0 . \tag{49}$$

Multiplying now the second relation (48) by a_2 and the relation (49) by $-3a_3$ and summing, we find

$$\left(2a_2^2 - 6a_1a_3\right)w_1 + a_1a_2 - 9a_0a_3 = 0 . \tag{50}$$

Returning to the notations made before and keeping into account that $w_1 = w_0$, one gets

$$J_z\dot{\psi}_0\dot{\varphi}_0 + \left(J_z - J_x\right)\dot{\psi}_0^2 \cos\theta_0 = mg\zeta , \tag{51}$$

where $\dot{\psi}_0$ and $\dot{\varphi}_0$ are the initial values for $\dot{\psi}$ and $\dot{\varphi}$, respectively.
The condition (51) is the condition for the regular precession. From the system (36) one obtains

$$\dot{\psi} = \frac{D_2 - A_2w}{1 - w^2},\ \dot{\varphi} = \omega_0 - \frac{D_2 - A_2w}{1 - w^2}w . \tag{52}$$

Moreover, since $w = w_0 = \mathrm{const}$ in the case of the regular precession, one deduces that $\dot{\psi} = \mathrm{const}$, $\dot{\varphi} = \mathrm{const}$, for $w^2 \neq 1$ ($\theta \neq 0$, $\theta \neq \pi$).

KINETIC ENERGY RECOVERY SYSTEM

The system is drawn in Fig. 2. It contains an axle of length l, radius r and density d_1; on the axle is put a flywheel of radii r and R, width h and density d_2. The flywheel is situated at the distance l_0 relative to the end O of the axle.

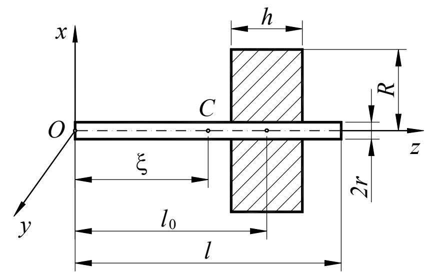

Figure 2. System KERS.

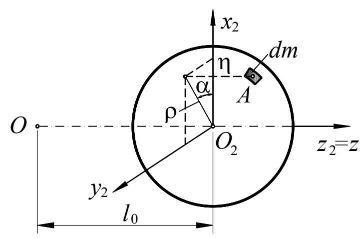

Figure 3. Moments of inertia for the flywheel.

We want to determine the moments of inertia relative to the point O.
By symmetry reasons the axes Ox, Oy, Oz are principal axes of inertia such that

$$J_{xy} = J_{xz} = J_{yz} = 0. \tag{53}$$

For the axle we have

$$J_x^{(1)} = J_y^{(1)} = \frac{\pi d_1 r^2 l^3}{3}, \tag{54}$$

$$J_z^{(1)} = \frac{\pi d_1 l r^4}{2}. \tag{55}$$

Denoting by O_2 the center of the flywheel, considering that the axes O_2x_2, O_2y_2 and O_2z_2 are parallel to the axes Ox, Oy and Oz, respectively, and taking an arbitrary point A of mass $\mathrm{d}m$ in the flywheel, we may write its coordinates as

$$x_A = \rho\cos\alpha,\ y_A = \rho\sin\alpha,\ z_A = l_0 + \eta, \tag{56}$$

wherefrom

$$x_A^2 = \rho^2\cos^2\alpha = \rho^2\frac{1+\cos(2\alpha)}{2},\ y_A^2 = \rho^2\sin^2\alpha = \rho^2\frac{1-\cos(2\alpha)}{2},\ z_A^2 = l_0^2 + \eta^2 + 2l_0\eta. \tag{57}$$

We have

$$\begin{aligned} J_x^{(2)} &= \int_{\mathrm{D}} \left(y_A^2 + z_A^2\right)\mathrm{d}m = d_2\int_0^{2\pi}\int_{l_0-\frac{h}{2}}^{l_0+\frac{h}{2}}\int_r^R \rho\left\{\frac{\rho^2}{2}\left[1-\cos(2\alpha)\right] + l_0^2 + \eta^2 + 2l_0\eta\right\}\mathrm{d}\rho\mathrm{d}\eta\mathrm{d}\alpha \\ &= \frac{\pi\left(R^2 - r^2\right)hd_2}{4}\left(R^2 + r^2 + 24l_0^2 + h^2\right), \end{aligned} \tag{58}$$

$$J_y^{(2)} = J_x^{(2)} = \frac{\pi\left(R^2 - r^2\right)hd_2}{4}\left(R^2 + r^2 + 24l_0^2 + h^2\right),\ J_z^{(2)} = \frac{\pi\left(R^2 - r^2\right)hd_2}{2}\left(R^2 + r^2\right);$$

hence

$$J_x = J_x^{(1)} + J_x^{(2)} = \frac{\pi d_1 r^2 l^3}{3} + \frac{\pi d_2\left(R^2 - r^2\right)h}{4}\left(R^2 + r^2 + 24l_0^2 + h^2\right), \tag{59}$$

$$J_z = J_z^{(1)} + J_z^{(2)} = \frac{\pi d_1 l r^4}{2} + \frac{\pi d_2\left(R^4 - r^4\right)h}{2}. \tag{60}$$

The position of the center of weight is given by

$$\zeta = \frac{\pi d_1 r^2 \frac{l^2}{2} + \pi d_2\left(R^2 - r^2\right)hl_0}{\pi d_1 r^2 l + \pi d_2\left(R^2 - r^2\right)h}. \tag{61}$$

HORIZONTAL KERS. REGULAR PRECESSION

In this case $\theta = \frac{\pi}{2}$, $w_0 = 0$ and the expressions (51) and (52) offer

$$J_z\dot{\psi}_0\dot{\varphi}_0 = mg\zeta, \tag{62}$$

$$\dot{\varphi} = \omega_0 = \text{const}, \tag{63}$$

$$\dot{\psi} = D_2 = \text{const}. \tag{64}$$

Keeping into account the relations (60) and (61), from the equation (62) we obtain

$$\frac{\dot{\psi}_0\omega_0}{2}\left[d_1 l r^4 + d_2\left(R^4 - r^4\right)h\right] = g\left[d_1 r^2\frac{l^2}{2} + d_2\left(R^2 - r^2\right)hl_0\right]. \tag{65}$$

Taking

$$l_0 = \frac{l}{2}, \tag{66}$$

the expression (65) becomes

$$\dot{\psi}_0\omega_0\left[d_1 l r^4 + d_2\left(R^4 - r^4\right)h\right] = g\left[d_1 r^2 l^2 + d_2\left(R^2 - r^2\right)hl\right] \tag{67}$$

or

$$\dot{\psi}_0 = \frac{gl}{\omega_0 r^2}\,\frac{1 + \dfrac{d_2}{d_1}\left(\dfrac{R^2}{r^2} - 1\right)\dfrac{h}{l}}{1 + \dfrac{d_2}{d_1}\left(\dfrac{R^4}{r^4} - 1\right)\dfrac{h}{l}}\,. \tag{68}$$

Denoting

$$\frac{d_2}{d_1} = \delta\,,\ \frac{R}{r} = \rho\,,\ \frac{h}{l} = \chi\,, \tag{69}$$

one obtains the relation

$$\dot{\psi}_0 = \frac{gl}{\omega_0 r^2}\,\frac{1 + \delta\left(\rho^2 - 1\right)\chi}{1 + \delta\left(\rho^4 - 1\right)\chi} \tag{70}$$

or

$$\frac{\omega_0 r^2\dot{\psi}_0}{gl} = \frac{1 + \delta\left(\rho^2 - 1\right)\chi}{1 + \delta\left(\rho^4 - 1\right)\chi} = ff\left(\delta, \rho, \chi\right). \tag{71}$$

A few diagram of variations are given in the next figures.

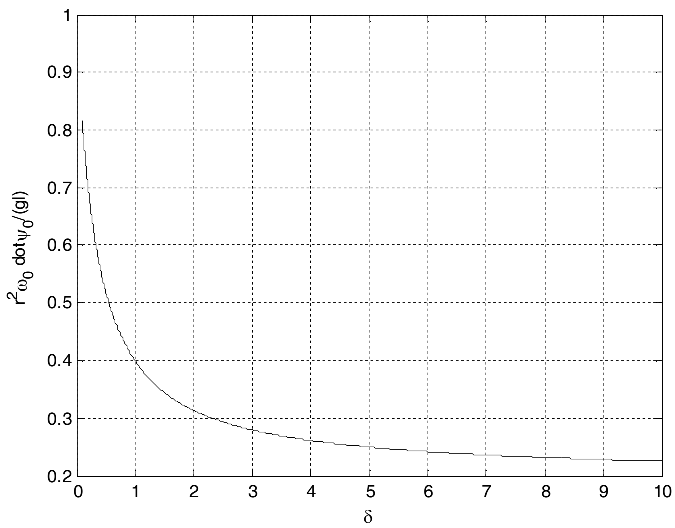

Figure 4. The variation $ff\left(\delta,\rho,\chi\right)$ for $\rho = 2$, $\chi = 0.2$, $1 \le \delta \le 10$

NUMERICAL EXAMPLE

We consider the values $R = 0.1\,\mathrm{m}$, $h = 0.05\,\mathrm{m}$, $r = 0.064\,\mathrm{m}$, $l = 0.06\,\mathrm{m}$, $d_1 = 7.8 \cdot 10^3\ \mathrm{kg/m^3}$, $d_2 = 2.7 \cdot 10^3\ \mathrm{kg/m^3}$, $l_0 = 0.3\,\mathrm{m}$, $\omega_0 = 2\pi\dfrac{60000}{60}\,\mathrm{rad/s}$, $g = 9.8065\,\mathrm{m/s^2}$.

One successively obtains $J_x = 0.031137\,\mathrm{kgm^2}$, $J_z = 0.029981\,\mathrm{kgm^2}$, $\zeta = 0.03\,\mathrm{m}$, $m = 8.52618\,\mathrm{kg}$, $\dot{\psi}_0 = 0.013316\,\mathrm{rad/s} = 0.763\ {}^0/\mathrm{s}$.

The kinetic energy reads now $T = \dfrac{1}{2}J_x\dot{\psi}^2 + \dfrac{1}{2}J_z\dot{\varphi}^2 = 591801\,\mathrm{J}$.

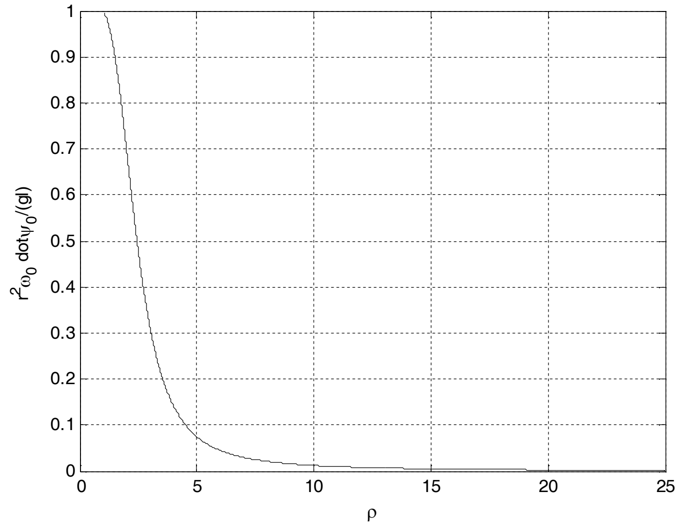

Figure 5. The variation $ff(\delta, \rho, \chi)$ for $\delta = 0.2$, $\chi = 0.2$, $1 \le \rho \le 25$.

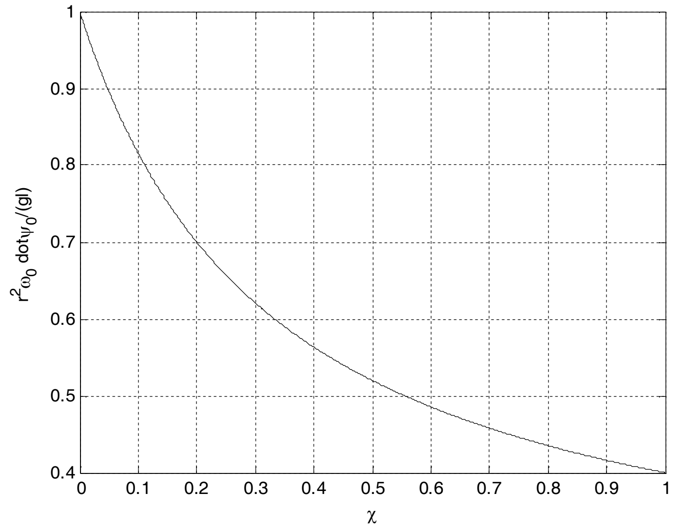

Figure 6. The variation $ff(\delta, \rho, \chi)$ for $\rho = 2$, $\delta = 0.2$, $0 \le \chi \le 1$

CONCLUSIONS

In our paper we presented a model of KERS and we studied its motion starting from the rigid body with one fixed point in the Lagrange–Poisson case. We obtained the condition for the regular precession and we particularized this condition in the case of a horizontal case. Some diagrams of variation of the angular velocity of the regular precession depending of different parameters are plotted. The main part of the kinetic energy comes from the rotation about the Oz axis. In this paper is considered a general flywheel and it is offered a method for the calculation of the moments of inertia. This method and the results obtained may be used for flywheels of similar shapes.

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